Bifurcating bright and dark solitary waves of the nearly nonlinear cubic-quintic Schrödinger equation[1]

Todd Kapitula[2]
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
USA

February 9, 2008

[1]1991 Mathematics Subject Classification: 34A26, 34C35, 34C37, 34C40, 34D15, 58F10, 78A60

[2]Current address: Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA

ABSTRACT. The existence of bright and dark multi-bump solitary waves for Ginzburg-Landau type perturbations of the cubic-quintic Schrödinger equation is considered. The waves in question are not perturbations of known analytic solitary waves, but instead arise as a bifurcation from a heteroclinic cycle in a three-dimensional ODE phase space. Using geometric singular perturbation techniques, regions in parameter space for which 1-bump bright and dark solitary waves will bifurcate are identified. The existence of N-bump dark solitary waves ($N \geq 1$) is shown via an application of the Exchange Lemma with Exponentially Small Error. N-bump bright solitary waves are shown to exist as a consequence of the work of Kapitula and Maier-Paape [**?**].

1 Introduction

The nonlinear cubic-quintic Schrödinger equation is given by

$$iA_t = A_{xx} + |A|^2 A + \alpha |A|^4 A, \tag{1.1}$$

where A is a complex-valued function of the variables $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. When $\alpha = 0$, the equation becomes the focusing cubic nonlinear Schrödinger equation, and is used to describe the propagation of the envelope of a light pulse in an optical fiber which has a Kerr-type nonlinear refractive index. For short pulses and high input peak pulse power the refractive index cannot be described by a Kerr-type nonlinearity, as the index is then influenced by higher-order nonlinearities. In materials with high nonlinear coefficients, such as semiconductors, semiconductor-doped glasses, and organic polymers, the saturation of the nonlinear refractive-index change is no longer neglible at moderately high intensities and should be taken into account ([**?**]). Equation (1.1) is the correct model to describe the propagation of the envelope of a light pulse in dispersive materials with either a saturable or higher-order refraction index ([**?**], [**?**]).

Equation (1.1) cannot really be thought of as a small perturbation of the cubic nonlinear Schrödinger equation, as it has been shown that a physically realistic value for the parameter α is $|\alpha| \sim 0.1$ ([**?**]). It turns out that the most physically interesting behavior occurs when the nonlinearity is saturating, i.e., $\alpha < 0$, so for the rest of this paper it will be assumed that $\alpha \sim -0.1$ ([**?**], [**?**], [**?**], [**?**], [**?**]). An optical fiber which satisfies this condition can be constructed, for example, by doping with two appropriate materials ([**?**]).

One of the more physically interesting phenomena associated with the double-doped optical fiber is the existence of bright solitary wave solutions ($|A(x)| \to 0$ as $|x| \to \infty$) in which the peak amplitude becomes a two-valued function of the pulse duration. These solutions have two different peak powers, and were proven to be stable ([**?**], [**?**], [**?**], [**?**]). The solution with the lower peak power corresponds to a perturbation of the one that exists for the cubic nonlinear Schrödinger equation, while the one with the larger peak power is due to the saturating nonlinearity.

Equation (1.1) describes an idealized fiber; therefore, it is natural to consider the perturbative PDE

$$iA_t = (1 + i\epsilon a_1)A_{xx} + i\epsilon\sigma A + (1 + i\epsilon d_1)|A|^2 A + (\alpha + i\epsilon d_2)|A|^4 A, \tag{1.2}$$

where $0 < \epsilon \ll 1$ and the other parameters are real and of $O(1)$. The parameter a_1 describes spectral filtering, σ describes the linear gain or loss due to the fiber, and d_1 and d_2 describe the nonlinear gain or loss due to the fiber. So that (1.2) is a well-defined PDE for $\epsilon > 0$, it will henceforth be assumed that $a_1 > 0$.

Solitary wave solutions to (1.2) are found by setting

$$A(x,t) = A(x)e^{i\mu t}, \tag{1.3}$$

and then finding homoclinic solutions for the ODE

$$(1 + i\epsilon a_1)A'' + (\mu + i\epsilon\sigma)A + (1 + i\epsilon d_1)|A|^2 A + (\alpha + i\epsilon d_2)|A|^4 A = 0, \tag{1.4}$$

where $' = d/dx$. Multiplying the above by $1 - i\epsilon a_1$ and setting

$$x = (1 + \epsilon^2 a_1^2)\hat{x}$$

yields the equivalent ODE

$$\begin{aligned} A'' + ((\mu + \epsilon^2 a_1\sigma) + i\epsilon(\sigma - a_1\mu))A + ((1 + \epsilon^2 a_1 d_1) + i\epsilon(d_1 - a_1))|A|^2 A \\ +((\alpha + \epsilon^2 a_1 d_2) + i\epsilon(d_2 - a_1\alpha))|A|^4 A = 0, \end{aligned} \tag{1.5}$$

where now $' = d/d\hat{x}$.

To simplify matters, all the $O(\epsilon^2)$ terms in the above equation will be dropped. This step is taken only so that the notational complexity is made as simple as possible, and can clearly be done without any loss of generality, as for ϵ small these terms are negligible. Upon dropping these terms, the ODE that will be studied can finally be written as

$$A'' + (\mu + i\epsilon(\sigma - a_1\mu))A + (1 + i\epsilon(d_1 - a_1))|A|^2 A + (\alpha + i\epsilon(d_2 - a_1\alpha))|A|^4 A = 0. \tag{1.6}$$

Equation (1.6) has been extensively studied by many authors ([?], [?], [?], [?], [?], [?], [?]). These papers have been concerned with finding various types of solutions, including fronts (kinks), bright solitary waves, and dark solitary waves. The methods employed have been both geometric ([?], [?], [?], [?]) and analytic ([?], [?], [?]).

Bright solitary waves exist when there are solutions to (1.6) which are homoclinic to $|A| = 0$. When $\epsilon = 0$ with

$$\frac{1}{4\alpha} < \mu < \frac{3}{16\alpha}, \tag{1.7}$$

one can find a bright solitary wave with a peak power larger than that associated with the cubic nonlinear Schrödinger equation. When $\epsilon = 0$ and μ is in the range

$$\frac{3}{16\alpha} < \mu < 0 \tag{1.8}$$

there exists dark solitary wave solutions. The dark solitary waves are solutions to (1.6) which have the property that $|A(x)| \to A_0 \neq 0$ as $|x| \to \infty$.

It can be hypothesized that for the perturbative PDE (1.2) there may be a competition between the bright and dark solitary waves for μ sufficiently close to the critical value

$$\mu^* = \frac{3}{16\alpha}. \tag{1.9}$$

It may be further hypothesized that for μ sufficiently near μ^* it may be possible to construct novel solutions by gluing together the bright and dark solitary waves in some way. The goal of this paper is to explore this possibility.

Doelman [?] has recently considered the problem of finding N-circuit solutions. These N-circuits are constructed by piecing together N copies of a dark solitary wave, and exist

as a solution to (1.6) for ϵ sufficiently small and the parameters in a certain domain in parameter space. Recent work by De Bouard [?] has shown that the dark solitary waves are an unstable solution to (1.1), due to the presence of a real positive eigenvalue for the operator obtained by linearizing about the wave. By a result of Alexander and Jones [?], since the original solitary wave has an unstable eigenvalue it is expected that the N-circuit solution will have at least N unstable eigenvalues.

It was previously stated that the bright solitary wave is a stable solution to (1.1). However, recent numerical work by Soto-Crespo et al [?] shows that this wave becomes an unstable solution to (1.2) for ϵ nonzero. The numerics suggest that this instability arises from the presence of a real eigenvalue for the linearized problem bifurcating out of the origin and into the right-half of the complex plane.

Due to these results, when attempting to construct stable solitary waves to (1.2) one would hope to avoid using either the bright or dark waves which exist for (1.1). This may by possible by considering the situation when $\mu = \mu^*$. When $\epsilon = 0$ and μ is equal to this critical parameter, there exists a front solution (kink) to (1.6) which has the asymptotic behavior

$$|A(x)| \to \begin{cases} 0, & x \to -\infty \\ A_0, & x \to \infty. \end{cases}$$

Furthermore, since (1.6) is invariant under $x \to -x$, there also exists a solution which satisfies

$$|A(x)| \to \begin{cases} A_0, & x \to -\infty \\ 0, & x \to \infty. \end{cases}$$

Thus, in the ODE phase space there exists a heteroclinic loop, from which solitary waves may bifurcate as ϵ is made nonzero. By following the proof in De Bouard [?] it can be conjectured that for the kink solution the linearized operator possesses no unstable eigenvalues. Therefore, it may be possible that any solitary waves bifurcating out of the heteroclinic loop may also not have any unstable eigenvalues.

Based upon the above discussion, it will be of interest to study the ODE (1.6) when μ is (at least) $O(\epsilon)$ close to μ^*. In particular, it will be of interest to study the dynamics of the ODE near the heteroclinic cycle for ϵ nonzero. Towards this end, it will first be shown that the cycle persists for ϵ nonzero for $\mu = \mu(\epsilon)$, with $\mu(\epsilon) \to \mu^*$ as $\epsilon \to 0$. After then fixing ϵ the dynamics will be studied as μ bifurcates from $\mu(\epsilon)$. It is important to note that μ is a natural bifurcation parameter in this problem, as it is a free parameter (recall equation (1.3)).

This paper will be devoted to proving the following theorems. Set

$$A_0^2 = -\frac{3}{4\alpha},$$

and define

$$\begin{aligned} \tilde{d}_1 &= d_1 - a_1 \\ \tilde{d}_2 &= d_2 - \alpha a_1 \\ \tilde{\sigma} &= -(\sigma + A_0^2 d_1 + A_0^4 d_2)/\epsilon. \end{aligned}$$

Assume that $\tilde{\sigma} = O(1)$ for all $\epsilon > 0$.

Before the main theorems can be stated, a preliminary lemma is needed regarding the persistence of the kink solitary wave for $\epsilon \neq 0$.

Lemma 1.1 *For $0 \leq \epsilon \ll 1$ there exists a $\mu(\epsilon)$, with $|\mu(\epsilon) - \mu^*| = O(\epsilon^2)$, such that a kink solitary wave exists.*

Remark 1.2 *Since (1.6) is invariant under the transformation $x \to -x$, the above lemma guarantees the existence of a heteroclinic cycle in the ODE phase space for ϵ nonzero.*

Theorem 1.3 *Let $0 < \epsilon \ll 1$, and assume that*

$$(\tilde{d}_1 + \frac{3}{2}A_0^2\tilde{d}_2)\tilde{\sigma} < 0.$$

There exists a $\mu_h(\epsilon) < \mu(\epsilon)$, with $\mu(\epsilon) - \mu_h(\epsilon) = O(e^{-c/\epsilon})$, such that a bright solitary wave exists. Furthermore, for each $N \geq 2$ there exists a bi-infinite sequence $\{\mu_k^N\}$ such that when $\mu = \mu_k^N$ there is an N-pulse solution to (1.6). The N-pulse is even in x. In addition,

$$|\mu_k^N - \mu_h(\epsilon)| = O(e^{-c|k|/\epsilon})$$

as $|k| \to \infty$.

Remark 1.4 *An N-pulse solution is constructed by piecing together N copies of a bright solitary wave.*

Remark 1.5 *The work of Soto-Crespo et al [**?**] suggests that the 1-pulse solution is indeed stable as a solution to (1.2).*

Theorem 1.6 *Suppose that $0 < \epsilon \ll 1$, and that $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 3$. Further assume that the parameters satisfy*

$$\begin{array}{l}(\tilde{d}_1 + \frac{3}{2}A_0^2\tilde{d}_2)\tilde{\sigma} < 0 \\ (\tilde{d}_1 + 2A_0^2\tilde{d}_2)\tilde{\sigma} < 0 \\ (\tilde{d}_1 + A_0^2\tilde{d}_2)(\tilde{d}_1 + \beta\tilde{d}_2) > 0,\end{array}$$

where

$$\beta = A_0^2 + \frac{1}{4}A_0^2(-\ln\frac{\eta}{A_0})^{-1} + O(\eta^2)$$

for $0 < \epsilon \ll \eta \ll 1$. Then there exists an $N(\epsilon) > 1$, with $N(\epsilon) \to \infty$ as $\epsilon \to 0$, such that N-circuit solutions exist for $1 \leq N < N(\epsilon)$.

Remark 1.7 *The suitable domain in parameter space is shown in Figure 7. In this figure the tildes have been dropped, $\beta = \alpha_2/\alpha_1$, and $x_0 = A_0$.*

The rest of the paper is organized as follows. In Section 2 the relevant three-dimensional ODE is set up, and the known solution structure is discussed. In Section 3 the relevant manifolds in the phase space are described. The bright, dark, and kink solitary waves are formed by intersecting the appropriate manifolds. This section is the most technical, and can be skipped on a first reading. Sections 4 and 5 discuss the intersection of these manifolds. In Section 6 the final proofs of the main theorems are given.

Acknowledgements: I would like to thank Alejandro Aceves for introducing me to the cubic-quintic Schrödinger equation and some of the references pertaining to it.

2 Setup of equations

stable bright solitary waves are possible for μ near μ^*. They do not consider the case of dark solitary waves in their paper, however.

In this section the first order system of ODEs that will be studied will be derived. The goal will be to write the system as compactly as possible so that the notational burden of the succeeding sections will be minimized. Define

$$\begin{aligned}\lambda(r) &= \mu^* + r^2 + \alpha r^4 \\ \omega(r) &= \tilde{d}_1 r^2 + \tilde{d}_2 r^4,\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\tilde{d}_1 &= d_1 - a_1 \\ \tilde{d}_2 &= d_2 - a_1\alpha,\end{aligned} \tag{2.2}$$

and furthermore set

$$\begin{aligned}\tilde{\sigma} &= -(\sigma - a_1\mu) \\ \mu &= \mu^* - \tilde{\mu}\end{aligned} \tag{2.3}$$

(μ^* is defined in (1.9)). After setting

$$A(\hat{x}) = r(\hat{x})e^{i\int^{\hat{x}}\phi(s)\,ds}$$

and substituting this expression into (1.6) one gets the third order system

$$\begin{aligned}r' &= ru \\ u' &= -u^2 - \lambda(r) + \phi^2 + \tilde{\mu} \\ \phi' &= -2u\phi + \epsilon(\omega(r) - \tilde{\sigma}).\end{aligned} \tag{2.4}$$

In the above equation the variable u is defined by $u = r'/r$.

The above equations will now be simplified notationally. First, redefine the variables $(\hat{x}, r, u, \phi)$ by

$$(t, x, y, z) = (\hat{x}, r, u, \phi). \tag{2.5}$$

Next, remove the tildes from all the variables. The ODE (2.4) then becomes

$$\begin{aligned}x' &= xy \\ y' &= -y^2 + z^2 - \lambda(x) + \mu \\ z' &= -2yz + \epsilon(\omega(x) - \sigma),\end{aligned} \tag{2.6}$$

where now $' = d/dt$ and $\omega(x) = d_1x^2 + d_2x^4$. For the subsequent analysis it will be useful to set

$$\sigma = \omega(x_0) + \epsilon\tilde{\sigma},$$

where x_0 is the largest positive root of $\lambda(x) = 0\,(x_0^2 = -3/4\alpha)$. Upon substituting the above expression into (2.6) and dropping the tilde, one obtains the equation

$$\begin{aligned}x' &= xy \\ y' &= -y^2 + z^2 - \lambda(x) + \mu \\ z' &= -2yz + \epsilon(\omega(x) - \omega(x_0) - \epsilon\sigma).\end{aligned} \tag{2.7}$$

Equation (2.7) is the one which will be studied in the subsequent sections. However, it is important to notice that the equation has the symmetry given in the following proposition. This symmetry will be used extensively in the upcoming sections.

Proposition 2.1 *Equation (2.7) remains invariant under*

$$(t, x, y, z) \to (-t, x, -y, -z).$$

In order to understand the behavior described in the succeeding sections, it is first necessary to understand the dynamics of (2.7) when $\epsilon = 0$. In Figure 1 the flow on the invariant (when $\epsilon = 0$) plane $\{z = 0\}$ is shown for different values of μ. When $\mu < 0$ there exists a heteroclinic connection which connects the invariant plane $\{x = 0\}$ to itself. This connection corresponds to a bright solitary wave. For $\mu = 0$ there exists a connection between $x = 0$ and $x = x_0$. This solutions corresponds to a kink. Due to the symmetry described in Proposition 2.1, there also exists a connection between x_0 and zero. Finally, when $\mu > 0$ there exists a solution connecting x_0 to itself, which corresponds to a dark solitary wave. Although it is not shown in the figure, this solution is actually embedded in the intersection of two two-dimensional manifolds in the phase space.

In the upcoming sections it will occasionally be useful to have an analytic expression for the front (kink) which exists when $\epsilon = \mu = 0$. This will be especially true when attempting to gain information on coefficients used in asymptotic expansions. Fortunately, due to Marcq et al [**?**] such an expression is available.

Proposition 2.2 *When $\epsilon = \mu = 0$ there exists a solution $(x(t), y(t), z(t)) = (\rho(t), u(t), \phi(t))$ to (2.7) which is given by*

$$\begin{aligned} \rho(t) &= \frac{x_0}{\sqrt{2}}(1 + \tanh(\frac{x_0}{2}t))^{1/2} \\ u(t) &= \rho'(t)/\rho(t) \\ \phi(t) &= 0. \end{aligned}$$

In the above x_0 is the largest positive root of $\lambda(x) = 0$ and is given by

$$x_0^2 = -\frac{3}{4\alpha}$$

(recall that $\alpha \sim -0.1$).

In the subsequent sections the notation $p \cdot t$ will be used to represent the trajectory of a point $p = (x, y, z)$ under the flow generated by (2.7). Under this convention, $p \cdot 0 = p$. In addition, given a point $p = (x, y, z)$, if it is of interest to only study the behavior of one of the variables, say x, under the flow, the notation $x \cdot t$ will be used.

When discussing the geometric objects in the following sections, the reader should consult Figure 2 to aid in the visualization.

3 Geometric objects

For this section assume that $\mu = \mu(\epsilon)$, with $\mu(\epsilon) = O(\epsilon^2)$. The validity of this assumption will be verified in Section 4.

3.1 Flow near $\{x = 0\}$

The goal of this subsection is to characterized the flow near the invariant plane $\{x = 0\}$. First, there exists a pair of critical points, say $(0, y_\pm(\epsilon), z_\pm(\epsilon))$, which satisfy the algebraic equations

$$\begin{aligned} -y^2 + z^2 - \lambda(0) + \mu(\epsilon) = 0 \\ -2yz - \epsilon(\omega(x_0) + \epsilon\sigma) = 0. \end{aligned} \tag{3.1}$$

It is not difficult to check that

$$y_\pm(0) = \pm\sqrt{-\lambda(0)}, \quad z_\pm(0) = 0;$$

furthermore, since $\mu(\epsilon) = O(\epsilon^2)$,

$$\begin{aligned} y_\pm(\epsilon) &= y_\pm(0) + O(\epsilon^2) \\ z_\pm(\epsilon) &= -\frac{\omega(x_0)}{2y_\pm(0)}\epsilon + O(\epsilon^2). \end{aligned} \tag{3.2}$$

It is also of interest to note that due to the symmetry described in Proposition 2.1, $y_- = -y_+ < 0$, $z_- = -z_+$. Furthermore, upon recalling the expression for x_0 given in Proposition 2.2, it is not difficult to see that $\lambda(0) = -x_0^2/4$, and hence $y_\pm(0) = \pm x_0/2$.

For convenience, set

$$\gamma_\pm = y_\pm(0), \quad \alpha_\pm = -\frac{\omega(x_0)}{2y_\pm(0)},$$

so that the critical points can be said to satisfy the asymptotic expansion

$$\begin{aligned} y_\pm(\epsilon) &= \gamma_\pm + O(\epsilon^2) \\ z_\pm(\epsilon) &= \alpha_\pm\epsilon + O(\epsilon^2). \end{aligned}$$

Linearizing the vector field about the critical points $(0, y_\pm, z_\pm)$ yields the matrix

$$A_\pm = \begin{pmatrix} y_\pm & 0 & 0 \\ 0 & -2y_\pm & 2z_\pm \\ 0 & -2z_\pm & -2y_\pm \end{pmatrix}, \tag{3.3}$$

which has the eigenvalues

$$\begin{aligned} \lambda_\pm^1 &= \gamma_\pm + O(\epsilon^2) \\ \lambda_\pm^2 &= -2\gamma_\pm + i\alpha_\pm\epsilon + O(\epsilon^2) \\ \lambda_\pm^3 &= \overline{\lambda_\pm^2}. \end{aligned} \tag{3.4}$$

Thus, the point $(0, y_+, z_+)$ has a one-dimensional unstable manifold, $W^u(0)$, coming out of the plane $\{x = 0\}$ tangent to the vector $(1, 0, 0)$, and a two-dimensional stable manifold embedded in the plane $\{x = 0\}$. In addition, the point $(0, y_-, z_-)$ has a one-dimensional stable manifold, $W^s(0)$, coming out of the invariant plane and tangent to $(1, 0, 0)$, and a two-dimensional unstable manifold embedded in the invariant plane.

Given $\delta > 0$, set

$$\mathcal{C}_0 = \{(x, y, z) : 0 \le x \le \delta, y = z = 0\}. \tag{3.5}$$

Due to the symmetry, this curve will be of paramount importance in subsequent sections. In the following proposition the term η is assumed to be positive and small.

Proposition 3.1 *Let $p = (x, y, z) \in \mathcal{C}_0$ be such that $x = O(\epsilon^n)$ for some $n \ge 1$. Then as long as $y_- + \eta \le y \cdot t \le y_+ - \eta$, $z \cdot t \ne 0$ for $t \ne 0$. Furthermore, $z \cdot (-t) = -z \cdot t$.*

Proof: By the nature of the function $\omega(x)$ it is clear that for $x = O(\epsilon^n)$, $\omega(x) = O(\epsilon^{2n})$. Since $x' = xy$, $x \cdot t = O(x \cdot 0)$ as long as $y_- + \eta \le y \cdot t \le y_+ - \eta$. Thus, if $x \cdot 0 = O(\epsilon^n)$, then $x \cdot t = O(\epsilon^n)$ for $y \cdot t$ in the prescribed range. By the above description of $\omega(x)$ for x small, this then implies that for $x \cdot 0 = O(\epsilon^n)$,

$$z' = -2yz - \epsilon(\omega(x_0) + \epsilon\sigma) + O(\epsilon^{2n+1}).$$

If $n \geq 1$ it is now clear that trajectories can cross the plane $\{z = 0\}$ at most once for $y_- + \eta \leq y \leq y_+ - \eta$. This proves the first part of the proposition. The second part follows immediately from the symmetry inherent in (2.7). ■

It is now of interest to determine the nature of the curve $\mathcal{C}_0$ as the flow carries it by the critical point $(0, y_+, z_+)$. Using the matrix A_+ defined in (3.3), the linear flow near this critical point satisfies

$$\begin{aligned} x' &= ax \\ y' &= -2ay + 2\epsilon bz \\ z' &= -2\epsilon bz - 2ay, \end{aligned} \tag{3.6}$$

where

$$a = \gamma_+ + O(\epsilon^2), \quad b = \alpha_+ + O(\epsilon).$$

Setting $y = r\sin\theta$, $z = r\cos\theta$ equation (3.6) can be rewritten as

$$\begin{aligned} x' &= ax \\ r' &= -2ar \\ \theta' &= \epsilon b, \end{aligned} \tag{3.7}$$

which has the solution

$$\begin{aligned} x(t) &= x_0 e^{at} \\ r(t) &= r_0 e^{-2at} \\ \theta(t) &= \theta_0 + \epsilon bt. \end{aligned} \tag{3.8}$$

For each $p \in \mathcal{C}_0$ define $t_0(p)$ to be such that

$$t_0(p) = \{\inf_{t>0} \,:\, p \cdot t \cap \{y_+ - r = \eta\} \neq \emptyset\}, \tag{3.9}$$

and set

$$\mathcal{C}_0 \cdot t_0 = \bigcup_{p \in \mathcal{C}_0} p \cdot t_0(p). \tag{3.10}$$

After translating the critical point $(0, y_+, z_+)$ to the origin, the curve $\mathcal{C}_0 \cdot t_0$ can be written parametrically as

$$\mathcal{C}_0 \cdot t_0 = \{(x, r, \theta) \,:\, x = x_1(s), r = \eta, \theta = -\pi/2 + \theta_1(s)\},$$

where $x_1(0) = 0$, $x_1'(s) > 0$, $x_1(s) \leq \eta$, and $\theta_1(s) = O(\epsilon)$.

It is of interest to determine the nature of $\mathcal{C}_0 \cdot t_0$ as the flow forces it to intersect the plane $\{x = \eta\}$. Given a point in $\mathcal{C}_0 \cdot t_0$, the time of flight, t_f, is defined by $x(t_f) = \eta$. Sustitution of this expression into (3.8) yields

$$t_f = \frac{1}{a} \ln \frac{\eta}{x_1(s)}. \tag{3.11}$$

This expression for t_f further yields that

$$\begin{aligned} r(t_f) &= \frac{1}{\eta} x_1^2(s) \\ \theta(t_f) &= -\frac{\pi}{2} + \theta_1(s) + \epsilon \frac{b}{a} \ln \frac{\eta}{x_1(s)}. \end{aligned} \tag{3.12}$$

The above curve is a logarithmic spiral centered upon the point at which $W^u(0)$ first intersects $\{x=\eta\}$. It is important to note that

$$\begin{array}{lcl} x_1(s)=O(\epsilon^n) & \Longrightarrow & \theta(t_f)=-\frac{\pi}{2}+\theta_1(s)+O(n\epsilon\ln\frac{1}{\epsilon}) \\ x_1(s)=O(e^{-c/\epsilon}) & \Longrightarrow & \theta(t_f)=-\frac{\pi}{2}+\theta_1(s)+O(1). \end{array} \tag{3.13}$$

Thus, the spiralling effect is seen only for those points which are initially exponentially close to the plane $\{x=0\}$.

Define the transverse sections to $W^u(0)$ and $W^s(0)$ by

$$\begin{array}{lcl} B_0^- & = & \{(x,y,z)\,:\,x=\eta, |y-y_-|\le\eta, |z-z_-|\le\eta\} \\ B_0^+ & = & \{(x,y,z)\,:\,x=\eta, |y-y_+|\le\eta, |z-z_+|\le\eta\}, \end{array} \tag{3.14}$$

where, as before, $\eta>0$ is small. Now define

$$\begin{array}{lclcl} p_0^s & = & W^s(0)\cap B_0^- & = & \{(\eta, y_-+O(\eta), z_-+O(\eta))\} \\ p_0^u & = & W^u(0)\cap B_0^+ & = & \{(\eta, y_++O(\eta), z_++O(\eta))\}, \end{array} \tag{3.15}$$

where the above represents the first intersection of the manifold with the set.

Assume that δ is sufficiently small so that for each $p\in\mathcal{C}_0$ there exists a $t>0$ such that $p\cdot t\cap B_0^+\neq\emptyset$. For each $p\in\mathcal{C}_0$ define

$$t_0^+(p)=\{\inf_{t>0}\,:\,p\cdot t\cap B_0^+\neq\emptyset\}, \tag{3.16}$$

and set $t_0^-(p)=-t_0^+(p)$. By using a symmetry argument it can be shown that $p\cdot t_0^-(p)\in B_0^-$. Finally, define the flow of $\mathcal{C}_0$ as it intersects the sets $B_0^\pm$ by

$$\begin{array}{lcl} \mathcal{C}_0\cdot t_0^- & = & \bigcup_{p\in\mathcal{C}_0} p\cdot t_0^-(p) \\ \mathcal{C}_0\cdot t_0^+ & = & \bigcup_{p\in\mathcal{C}_0} p\cdot t_0^+(p). \end{array} \tag{3.17}$$

When $\epsilon=0$ these sets are given by

$$\begin{array}{lcl} \mathcal{C}_0\cdot t_0^- & = & \{(x,y,z)\,:\,x=\eta, y_-\le y\le y_-+\eta, z=0\} \\ \mathcal{C}_0\cdot t_0^+ & = & \{(x,y,z)\,:\,x=\eta, y_+-\eta\le y\le y_+, z=0\}. \end{array} \tag{3.18}$$

The blow-up sets will be defined next. This set corresponds to points p such that $|p\cdot t|\to\infty$ in finite time ([**?**]). As a preliminary, set

$$\begin{array}{lcl} B_{0,s}^- & = & \{(x,y,z)\,:\,0\le x\le\eta, y=y_--\eta, |z-z_-|\le\eta\} \\ B_{0,s}^+ & = & \{(x,y,z)\,:\,0\le x\le\eta, y=y_++\eta, |z-z_+|\le\eta\}. \end{array} \tag{3.19}$$

Following Kapitula and Maier-Paape [**?**] there exist sets

$$\begin{array}{lcl} \mathcal{C}_{-\infty} & = & \{p\in B_{0,s}^-\,:\,\exists\, t(p)>0 \text{ such that } \lim_{t\to t^-(p)} p\cdot t=(0,-\infty,0)\} \\ \mathcal{C}_{+\infty} & = & \{p\in B_{0,s}^+\,:\,\exists\, t(p)<0 \text{ such that } \lim_{t\to t^+(p)} p\cdot t=(0,+\infty,0)\}. \end{array} \tag{3.20}$$

The above sets are smooth curves, and symmetry considerations yield that $(x,y,z)\in\mathcal{C}_{-\infty}$ implies that $(x,-y,-z)\in\mathcal{C}_{+\infty}$. For $p\in\mathcal{C}_{+\infty}$ set

$$t_{+\infty}(p)=\{\inf_{t>0}\,:\,p\cdot t\cap B_0^+\neq\emptyset\}, \tag{3.21}$$

and for $p \in \mathcal{C}_{-\infty}$ define

$$t_{-\infty}(p) = \{\sup_{t<0} \, : \, p \cdot t \cap B_0^- \neq \emptyset\}. \tag{3.22}$$

Finally, the flow of these sets will be given by

$$\begin{aligned} \mathcal{C}_{-\infty} \cdot t_{-\infty} &= \bigcup_{p \in \mathcal{C}_{-\infty}} p \cdot t_{-\infty}(p) \\ \mathcal{C}_{+\infty} \cdot t_{+\infty} &= \bigcup_{p \in \mathcal{C}_{+\infty}} p \cdot t_{+\infty}(p). \end{aligned} \tag{3.23}$$

When $\epsilon = 0$ these sets are given by

$$\begin{aligned} \mathcal{C}_{-\infty} \cdot t_{-\infty} &= \{(x, y, z) \, : \, x = \eta, y_- - \eta < y \leq y_-, z = 0\} \\ \mathcal{C}_{+\infty} \cdot t_{+\infty} &= \{(x, y, z) \, : \, x = \eta, y_+ < y \leq y_+ - \eta, z = 0\}. \end{aligned} \tag{3.24}$$

This subsection can now be concluded with the following lemma, whose conclusion follows from the above discussion. The fact that $\mathcal{C}_{-\infty} \cdot t_{-\infty}$ and $\mathcal{C}_{+\infty} \cdot t_{+\infty}$ are logarithmic spirals follows from the same argument as that which led to the description of $\mathcal{C}_0 \cdot t_0^-$ and $\mathcal{C}_0 \cdot t_0^+$.

Lemma 3.2 *Suppose that $\epsilon\omega(x_0) \neq 0$. Then $\mathcal{C}_0 \cdot t_0^-$ and $\mathcal{C}_{-\infty} \cdot t_{-\infty}$ are logarithmic spirals centered on p_0^s, and the curves $\mathcal{C}_0 \cdot t_0^+$ and $\mathcal{C}_{-\infty} \cdot t_{+\infty}$ are logarithmic spirals centered on p_0^u. The spiralling is seen only $O(e^{-c/\epsilon})$ close to the points p_0^s and p_0^u; furthermore, their location is described within $O(\epsilon)$ by equations (3.18) and (3.24).*

Remark 3.3 *By the symmetry described in Proposition 2.1 it is necessarily true that $(\eta, y, z) \in \mathcal{C}_0 \cdot t_0^-$ implies that $(\eta, -y, -z) \in \mathcal{C}_0 \cdot t_0^+$, and $(\eta, y, z) \in \mathcal{C}_{-\infty} \cdot t_{-\infty}$ implies that $(\eta, -y, -z) \in \mathcal{C}_{+\infty} \cdot t_{+\infty}$.*

3.2 Flow near $\mathcal{M}_\epsilon$

The purpose of this subsection is to describe the flow generated by (2.7) near the point $(x_0, 0, 0)$, where x_0 is the largest positive zero of $\lambda(x) = 0$.

When $\epsilon = 0$ there exists a critical manifold $\mathcal{M}_0$ which is given by

$$\mathcal{M}_0 = \{(x, y, z) \, : \, y = 0, \ z^2 - \lambda(x) = 0, \ 4z^2 + x\lambda'(x) < 0\}. \tag{3.25}$$

This manifold is normally hyperbolic, and therefore smoothly perturbs to a manifold $\mathcal{M}_\epsilon$ for ϵ sufficiently small. Furthermore, the manifold $\mathcal{M}_\epsilon$ has a two-dimensional stable manifold, $W^s(\mathcal{M}_\epsilon)$, and a two-dimensional unstable manifold, $W^u(\mathcal{M}_\epsilon)$, which are smooth perturbations of the center-stable and center-unstable manifolds which exist when $\epsilon = 0$ (Fenichel [?], Jones [?]).

As in the previous section, assume that $\mu = \mu(\epsilon) = O(\epsilon^2)$. The critical points for (2.7) which lie on $\mathcal{M}_\epsilon$ satisfy $y = 0$ and

$$\begin{gathered} \lambda(x) - z^2 + O(\epsilon^2) = 0 \\ \omega(x) - \omega(x_0) - \epsilon\sigma = 0. \end{gathered} \tag{3.26}$$

A Taylor expansion about the point x_0 yields that

$$x - x_0 = \epsilon\frac{1}{\omega'(x_0)} + O(\epsilon^2). \tag{3.27}$$

Substituting the expression in (3.27) into the first equation of (3.26) and taking a Taylor expansion yields

$$\epsilon\frac{\lambda'(x_0)}{\omega'(x_0)} - z^2 + O(\epsilon^2) = 0, \tag{3.28}$$

from which one gets that

$$z^2 = \epsilon\frac{\lambda'(x_0)}{\omega'(x_0)} + O(\epsilon^2). \tag{3.29}$$

This in turn yields

$$z = \pm\sqrt{\lambda'(x_0)\frac{\sigma}{\omega'(x_0)}}\epsilon^{1/2} + O(\epsilon). \tag{3.30}$$

Letting $(x^*(\epsilon), 0, \pm z^*(\epsilon)) \in \mathcal{M}_\epsilon$ represent the critical points, it is now seen that these points have the asymptotic expansion

$$\begin{aligned} x^*(\epsilon) &= x_0 + \frac{\sigma}{\omega'(x_0)}\epsilon + O(\epsilon^2) \\ z^*(\epsilon) &= \sqrt{\lambda'(x_0)\frac{\sigma}{\omega'(x_0)}}\epsilon^{1/2} + O(\epsilon). \end{aligned} \tag{3.31}$$

Since $\lambda'(x_0) < 0$, in order for the above expressions to make sense the following assumption must hold.

Proposition 3.4 *If the parameters d_1, d_2, and σ are chosen so that*

$$\omega'(x_0)\sigma < 0,$$

i.e.,

$$(d_1 + 2x_0^2 d_2)\sigma < 0,$$

then there exist critical points on $\mathcal{M}_\epsilon$ whose x and z components are given by (3.31).

The next goal is to get an asymptotic description of $\mathcal{M}_\epsilon$ within $O(\epsilon)$ of $\{z = 0\}$. A preliminary lemma is first needed. In the following, let the curve $\mathcal{C}_{x_0}$ be given by

$$\mathcal{C}_{x_0} = \{(x, y, z) : y = z = 0, |x - x_0| \leq \nu\}. \tag{3.32}$$

Lemma 3.5 *There exists an $\epsilon_0 > 0$ such that if $0 \leq \epsilon < \epsilon_0$, then $\mathcal{C}_{x_0} \cap \mathcal{M}_\epsilon \neq \emptyset$.*

Proof: Clearly, $\mathcal{M}_\epsilon$ intersects the plane $\{z = 0\}$ nontrivially. Set

$$\{(x_\epsilon, y_\epsilon, 0)\} = \mathcal{M}_\epsilon \cap \{z = 0\}.$$

In order to prove the lemma, it must be shown that $y_\epsilon = 0$ for $\epsilon \neq 0$. By the definition of $\mathcal{M}_0$ and the fact that $\mathcal{M}_\epsilon$ is a smooth perturbation of $\mathcal{M}_0$ it is clear that

$$\lim_{\epsilon \to 0} y_\epsilon = 0.$$

In a sufficiently small neighborhood of $\mathcal{M}_\epsilon$ the manifolds $W^s(\mathcal{M}_\epsilon)$ and $W^u(\mathcal{M}_\epsilon)$ satisfy the relation

$$W^s(\mathcal{M}_\epsilon) \cap W^u(\mathcal{M}_\epsilon) = \mathcal{M}_\epsilon.$$

This immediately yields that

$$W^s(\mathcal{M}_\epsilon) \cap W^u(\mathcal{M}_\epsilon) \cap \{z = 0\} = \{(x_\epsilon, y_\epsilon, 0)\}. \tag{3.33}$$

Suppose without loss of generality that

$$\lim_{t\to\pm\infty} (x_\epsilon, y_\epsilon, 0) \cdot t = (x^*, 0, \pm z^*).$$

By the symmetry of the ODE, if $(x_\epsilon, y_\epsilon, 0) \cdot t$ is a solution, then so is $(x_\epsilon, -y_\epsilon, 0) \cdot (-t)$. This immediately yields that

$$\lim_{t\to\pm\infty} (x_\epsilon, -y_\epsilon, 0) \cdot t = (x^*, 0, \mp z^*),$$

so that

$$\{(x_\epsilon, -y_\epsilon, 0)\} \subset W^s(\mathcal{M}_\epsilon) \cap W^u(\mathcal{M}_\epsilon) \cap \{z = 0\}.$$

But equation (3.33) then gives

$$(x_\epsilon, y_\epsilon, 0) = (x_\epsilon, -y_\epsilon, 0)$$

so that $y_\epsilon = 0$. ∎

For the rest of this subsection set $\zeta = |z| + \epsilon$. The slow manifold $\mathcal{M}_\epsilon$ is given by the graph

$$\mathcal{M}_\epsilon = \{(x, y, z, e) \,:\, x = \mathcal{M}_x(z, \epsilon), y = \mathcal{M}_y(z, \epsilon)\}. \tag{3.34}$$

By the definition of $\mathcal{M}_0$ the function $\mathcal{M}_y$ satisfies the relation $\mathcal{M}_y(z, 0) = 0$. Furthermore, the conclusion of Lemma 3.5 yields that $\mathcal{M}_y(0, \epsilon) = 0$. Thus, a Taylor expansion of $\mathcal{M}_y$ gives

$$\mathcal{M}_y(z, \epsilon) = \epsilon z \tilde{\mathcal{M}}_y(z, e), \tag{3.35}$$

where the function $\tilde{\mathcal{M}}_y$ is uniformly bounded for ζ sufficiently small. The following lemma gives a description of the functions $\mathcal{M}_x$ and $\tilde{\mathcal{M}}_y$.

Lemma 3.6 *The functions $\mathcal{M}_x$ and $\mathcal{M}_y$ comprising the graph of $\mathcal{M}_\epsilon$ have the Taylor expansion*

$$\begin{aligned}
\mathcal{M}_x(z, \epsilon) &= x_0 + \frac{1}{\lambda'(x_0)} z^2 + \tilde{x}\epsilon^2 + O(\zeta^3) \\
\mathcal{M}_y(z, \epsilon) &= \epsilon z(-\frac{2\sigma}{x_0 \lambda'(x_0)}\epsilon + O(\zeta^2)),
\end{aligned}$$

where $\tilde{x}$ is an $O(1)$ constant.

Proof: The basic idea of the proof is to write out a Taylor expansion for both $\mathcal{M}_x$ and $\mathcal{M}_y$ and to use the fact that the manifold $\mathcal{M}_\epsilon$ is invariant under the flow. After using (3.35) the functions can be expanded as

$$\begin{aligned}
\mathcal{M}_x(z, \epsilon) &= x_0 + x_1 z + x_2 \epsilon + x_3 z^2 + x_4 \epsilon z + x_5 \epsilon^2 + O(\zeta^3) \\
\mathcal{M}_y(z, \epsilon) &= \epsilon z(y_0 + y_1 z + y_2 \epsilon + O(\zeta^2)),
\end{aligned}$$

where x_0 is such that $\lambda(x_0) = 0$. In order to prove the lemma, it must be shown that

$$x_1 = x_2 = x_4 = y_0 = y_1 = 0$$

and

$$x_3 = \frac{1}{\lambda'(x_0)}, \; y_2 = -\frac{2\sigma x_3}{x_0}.$$

When $\epsilon = 0$ the function $\mathcal{M}_x$ satisfies the relation

$$z^2 - \lambda(\mathcal{M}_x(z, 0)) = 0.$$

Taking a Taylor expansion of $\lambda(x)$ about x_0 and equating coefficients quickly yields the desired relation for x_1 and x_3.

In order to show that $x_2 = 0$, consider the following argument. The flow on $\mathcal{M}_\epsilon$ satisfies

$$z' = -2\mathcal{M}_y(z,\epsilon)z + \epsilon(\omega(\mathcal{M}_x(z,\epsilon)) - \omega(x_0) - \epsilon\sigma), \tag{3.36}$$

so that when $z = 0$

$$z' = \epsilon((x_2\omega'(x_0) - \sigma)\epsilon + O(\epsilon^2)).$$

The above expression is arrived at upon taking a Taylor expansion of $\omega(x)$ about x_0 and using the expansion for $\mathcal{M}_x(0, \epsilon)$. Note that $z' = O(\epsilon^2)$. Now, the fact that on $\mathcal{M}_\epsilon$, $y = \mathcal{M}_y(z, \epsilon)$ implies that

$$y' = z'\partial_z\mathcal{M}_y(z, \epsilon).$$

Using the Taylor expansion for $\mathcal{M}_y$ and the fact that $z' = O(\epsilon^2)$ yields that when $z = 0$, $y' = O(\epsilon^3)$. But when $z = 0$,

$$y' = -\mathcal{M}_y^2(0,\epsilon) - \lambda(\mathcal{M}_x(0,\epsilon)) + \mu(\epsilon).$$

Since $\mu(\epsilon) = O(\epsilon^2)$, after taking a Taylor expansion of $\lambda(x)$ about x_0 one gets that

$$y' = -x_2\lambda'(x_0)\epsilon + O(\epsilon^2).$$

Thus, the fact that $y' = O(\epsilon^3)$ necessarily gives that $x_2 = 0$.

Using the above results and equation (3.36) gives that the equation for z on $\mathcal{M}_\epsilon$ is

$$z' = -\sigma\epsilon^2 + (x_3\omega'(x_0) - 2y_0)\epsilon z^2 + x_4\omega'(x_0)\epsilon^2 z + x_5\omega'(x_0)\epsilon^3 + O(\zeta^4).$$

Since $x' = xy$, on $\mathcal{M}_\epsilon$

$$\mathcal{M}_x' = \mathcal{M}_x\mathcal{M}_y. \tag{3.37}$$

But

$$\begin{aligned} \mathcal{M}_x' &= z'\partial_z\mathcal{M}_x \\ &= -2\sigma x_3\epsilon^2 z - \sigma x_4\epsilon^3 + O(\zeta^4) \end{aligned}$$

and

$$\mathcal{M}_x\mathcal{M}_y = x_0y_0\epsilon z + x_0y_1\epsilon z^2 + x_0y_2\epsilon^2 z + O(\zeta^4),$$

so that (3.37) becomes

$$-2\sigma x_3\epsilon^2 z - \sigma x_4\epsilon^3 + O(\zeta^4) = x_0y_0\epsilon z + x_0y_1\epsilon z^2 + x_0y_2\epsilon^2 z + O(\zeta^4).$$

Upon equating coefficients the desired result is reached. ∎

3.3 Flow near $\mathcal{M}_\epsilon$

Recall that the flow on $\mathcal{M}_\epsilon$ is given by (3.36). Using the result of Lemma 3.6 and Taylor expanding $\omega(x)$ around x_0 then yields that on $\mathcal{M}_\epsilon$ the flow is given by

$$z' = \epsilon(\frac{\omega'(x_0)}{\lambda'(x_0)}z^2 - \sigma\epsilon + O(\epsilon^2)) + O(\zeta^4). \tag{3.38}$$

Note that the above equation implies that for $|z| = O(\epsilon)$,

$$z' = -\epsilon^2\sigma + O(\epsilon^3). \tag{3.39}$$

This estimate will turn out to be crucial for the estimates that follow.

In order to determine the nature of the flow near $\mathcal{M}_\epsilon$, it will be desirous to use Fenichel coordinates ([?], [?], [?]). Before doing so, it will first be beneficial to understand the matrix arrived at upon linearizing about $\mathcal{M}_\epsilon$. Specifically, the manner in which the eigenvalues and eigenvectors vary as a function of z and ϵ must be understood.

Linearizing the flow (2.7) about a point $(x, y, z) \in \mathcal{M}_\epsilon$ yields the matrix

$$A_{\mathcal{M}_\epsilon} = \begin{pmatrix} y & x & 0 \\ -\lambda'(x) & -2y & 2z \\ \epsilon\omega'(x) & -2z & -2y, \end{pmatrix} \tag{3.40}$$

which has the characteristic equation

$$P(\gamma, z, \epsilon) = (\gamma - y)((\gamma + 2y)^2 + 4z^2) + x(\lambda'(x)(\gamma + 2y) - 2\epsilon\omega'(x)z) = 0. \tag{3.41}$$

The above equation has two solutions which are $O(1)$, say $\gamma^\pm(z, \epsilon)$, with $\gamma^+ = -\gamma^-$ and $\gamma^-(0, 0) = -\sqrt{-x_0\lambda'(x_0)}$.

In the following calculations the result of Lemma 3.6 will always be implicitly used. It is easy to check that

$$P_\gamma(\gamma^-, 0, 0) = -2x_0\lambda'(x_0)$$

and

$$P_z(\gamma^-, 0, 0) = P_\epsilon(0, 0) = 0,$$

so that

$$\gamma^-_z(0, 0) = \gamma^-_\epsilon(0, 0) = 0.$$

The above follows immediately from the fact that

$$\gamma^-_z = -P_z/P_\gamma, \quad \gamma^-_\epsilon = -P_\epsilon/P_\gamma.$$

Taking second derivatives gives

$$\begin{aligned} P_{zz}(\gamma^-, 0, 0) &= -8\sqrt{-x_0\lambda'(x_0)} \\ P_{z\epsilon}(\gamma^-, 0, 0) &= -2x_0\omega'(x_0). \end{aligned}$$

Since

$$\gamma_{zz} = -P_{zz}/P_\gamma, \quad \gamma_{z\epsilon} = -P_{z\epsilon}/P_\gamma,$$

this immediately gives that

$$\begin{aligned} \gamma^-_{zz}(0, 0) &= \frac{4}{\sqrt{-x_0\lambda'(x_0)}} \\ \gamma^-_{z\epsilon}(0, 0) &= -\frac{\omega'(x_0)}{\lambda'(x_0)}. \end{aligned}$$

The proof of the following proposition is now complete.

Proposition 3.7 *The $O(1)$ eigenvalues $\gamma^\pm$ of $A_{\mathcal{M}_\epsilon}$ satisfy*

$$\begin{array}{ll} 1) & \gamma^+(z,\epsilon) = -\gamma^-(z,\epsilon) \\ 2) & \gamma^-(z,\epsilon) = -\sqrt{-x_0\lambda'(x_0)} + \dfrac{2}{\sqrt{-x_0\lambda'(x_0)}}z^2 - \dfrac{\omega'(x_0)}{\lambda'(x_0)}\epsilon z + \tilde{\gamma}\epsilon^2 + O(\zeta^3), \end{array}$$

where $\tilde{\gamma}$ is an $O(1)$ constant.

Although the following information is not necessary at the moment, it will be useful at a later time. The eigenvector $\mathbf{v}_{\gamma^-}$ of $A_{\mathcal{M}_\epsilon}$ associated with the eigenvalue γ^- is given by

$$\mathbf{v}_{\gamma^-} = (x, \gamma^- - y, \frac{-2(\gamma^- - y)z + \epsilon x\omega'(x)}{\gamma^- + 2y})^T. \tag{3.42}$$

Let $\mathbf{v}_{\gamma^-} = (v_1, v_2, v_3)^T$. After using the expansions given in Lemma 3.6 and Proposition 3.7 the following proposition is realized.

Proposition 3.8 *The components of the eigenvector $\mathbf{v}_{\gamma_-}$ are given by the expansions*

$$\begin{array}{rcl} v_1 & = & x_0 + \dfrac{1}{\lambda'(x_0)}z^2 + \tilde{x}\epsilon^2 + O(\zeta^3) \\ v_2 & = & -\sqrt{-x_0\lambda'(x_0)} + \dfrac{2}{\sqrt{-x_0\lambda'(x_0)}}z^2 - \dfrac{\omega'(x_0)}{\lambda'(x_0)}\epsilon z + \tilde{\gamma}\epsilon^2 + O(\zeta^3) \\ v_3 & = & -2z - \dfrac{x_0\omega'(x_0)}{\sqrt{-x_0\lambda'(x_0)}}\epsilon + O(\zeta^2), \end{array}$$

where $\tilde{x}$ and $\tilde{\gamma}$ are $O(1)$ constants.

Remark 3.9 *Due to the symmetry of the ODE, the eigenvector associated with γ^+ is such that the second and third components are the negative of those given above.*

Remark 3.10 *When the above expressions are evaluated at the critical points $(x^*(\epsilon), 0, \pm z^*(\epsilon))$, it is of interest to note that the third component, v_3, satisfies the estimate*

$$v_3 = -2(\pm z^*(\epsilon)) + O(\epsilon).$$

The above expression is valid because it is known that $z^*(\epsilon) = O(\epsilon^{1/2})$.

Recall that for $|z| = O(\epsilon)$ that the flow on $\mathcal{M}_\epsilon$ satisfies

$$z' = -\epsilon^2\sigma + O(\epsilon^3).$$

Thus, in Fenichel coordinates the equations near $\mathcal{M}_\epsilon$ for $|z| = O(\epsilon)$ may be written as

$$\begin{array}{rcl} a' & = & \Lambda(a,b,z,\epsilon)a \\ b' & = & \Gamma(a,b,z,\epsilon)b \\ z' & = & \epsilon^2(-\sigma + h(a,b,z,\epsilon)ab). \end{array} \tag{3.43}$$

In the above equation the set $a = 0$ refers to $W^s(\mathcal{M}_\epsilon)$, the set $b = 0$ refers to $W^u(\mathcal{M}_\epsilon)$, and the function h is uniformly bounded. Using the eigenvalue expansion in Proposition 3.7 one can say that for $|z| = O(\epsilon)$,

$$\Lambda(0,0,z,\epsilon) = \sqrt{-x_0\lambda'(x_0)} + O(\epsilon^2), \quad \Gamma(0,0,z,\epsilon) = -\Lambda(0,0,z,\epsilon).$$

After taking a Taylor expansion the equations (3.43) may then be rewritten as

$$\begin{aligned} a' &= \sqrt{-x_0\lambda'(x_0)}\,a + O((|a|+|b|+\epsilon)^2) \\ b' &= -\sqrt{-x_0\lambda'(x_0)}\,b + O((|a|+|b|+\epsilon)^2) \\ z' &= \epsilon^2(-\sigma + h(a,b,z,\epsilon)ab). \end{aligned} \tag{3.44}$$

Define the set B_{x_0} by

$$B_{x_0} = \{(a,b,z)\,:\,|a|+|b|\le\nu, |z|\le O(\epsilon)\}. \tag{3.45}$$

For ϵ and ν sufficiently small it is clear that one can, without loss of generality, use the linear equations

$$\begin{aligned} a' &= \sqrt{-x_0\lambda'(x_0)}\,a \\ b' &= -\sqrt{-x_0\lambda'(x_0)}\,b \\ z' &= -\epsilon^2\sigma \end{aligned} \tag{3.46}$$

when discussing the flow in B_{x_0}. For convenience, in the rest of this subsection $\gamma = \sqrt{-x_0\lambda'(x_0)}$, so that (3.46) can be rewritten as

$$\begin{aligned} a' &= \gamma a \\ b' &= -\gamma b \\ z' &= -\epsilon^2\sigma. \end{aligned} \tag{3.47}$$

Now that the equations for the flow near $\mathcal{M}_\epsilon$ have been determined, it will be of interest to determine the nature of the set $\mathcal{C}_{x_0}$ as it exits B_{x_0} under the influence of the flow. To be precise, for each $p \in \mathcal{C}_{x_0}$ define t_{x_0} by

$$t_{x_0}(p) = \{\sup_{t<0}\,:\,p\cdot t \in B_{x_0}^+\},$$

and set

$$\mathcal{C}_{x_0}\cdot t_{x_0} = \bigcup_{p\in\mathcal{C}_{x_0}} p\cdot t_{x_0}(p). \tag{3.48}$$

The set $\mathcal{C}_{x_0}\cdot t_{x_0}$ will be close to the set $a=0$, i.e., close to $W^s(\mathcal{M}_\epsilon)$, as it exits $B_{x_0}^+$, where

$$B_{x_0}^+ = \{(x,y,z)\,:\,x = x_0-\nu, |y-(\sqrt{-\lambda'(x_0)/x_0})\nu|\le\nu, |z|\le\nu\} \tag{3.49}$$

The following lemma gives a determination as to how close.

Lemma 3.11 *Let $p\in\mathcal{C}_{x_0}$ be such that $t_{x_0}(p) = O(1/\epsilon)$. The curve $\mathcal{C}_{x_0}\cdot t_{x_0}$ is $C^1-O(e^{-c/\epsilon})$ close to $a=0$ in a neighborhood of $p\cdot t_{x_0}(p)$, where $c>0$ is some constant.*

Proof: In order to prove the lemma the time-reversed flow for (3.47) must be considered. After such a time reversal, the solution to (3.47) is given by

$$\begin{aligned} a(t) &= a_0 e^{-\gamma t} \\ b(t) &= b_0 e^{\gamma t} \\ z(t) &= z_0 + \epsilon^2\sigma t. \end{aligned} \tag{3.50}$$

In Fenichel coordinates the curve $\mathcal{C}_{x_0}$ can be parametrically represented as $(a_0(s), b_0(s), z_0(s)) = (s,s,0)$ with $0\le s\le\nu$. Substituting this set of initial conditions into (3.50) yields that the flow of $\mathcal{C}_{x_0}$ is governed by

$$\begin{aligned} a(t) &= se^{-\gamma t} \\ b(t) &= se^{\gamma t} \\ z(t) &= \epsilon^2\sigma t. \end{aligned} \tag{3.51}$$

A solution to (3.51) leaves B_{x_0} when $b(t) = \nu$. Solving this equation yields that the time-of-flight, t_f, is given by

$$t_f = \frac{1}{\gamma} \ln \frac{\nu}{s}.$$

Substituting t_f into (3.51) yields

$$\begin{aligned} a(t_f) &= \frac{s^2}{\nu} \\ z(t_f) &= -\epsilon^2 \frac{1}{\gamma} \ln \frac{\nu}{s}. \end{aligned} \tag{3.52}$$

The above equation is a parametric representation of a curve on the section $B^+_{x_0}$. It can be rewritten as

$$a(z) = \nu e^{\beta z/\epsilon^2},$$

where

$$\beta = 2\gamma/\sigma, \quad \sigma z < 0.$$

The conclusion of the lemma is now clear, as when $z = O(\epsilon)$ with $\sigma z < 0$,

$$a(z) = O(e^{-c/\epsilon}), \quad a'(z) = O(\frac{e^{-c/\epsilon}}{\epsilon^2}), \qquad \blacksquare.$$

4 $W^u(0) \cap W^s(\mathcal{M}_\epsilon)$ transversely

After appending $\mu' = 0$ and $\epsilon' = 0$ to (2.7) one arrives at a three-dimensional manifold $W^u(0)$ (abusing notation here) and a four-dimensional manifold $W^s(\mathcal{M}_\epsilon)$ in the five-dimensional phase space. Since $x' > 0$ along the relevant trajectory for $0 < x < x_0$, by continuity this holds for ϵ and μ sufficiently small. As such, for $0 < x < x_0$ the manifolds can be parameterized in the following manner:

$$\begin{aligned} W^u(0) &= \{(x,y,z) \,:\, y = \mathcal{U}^y(x,\mu,\epsilon), z = \mathcal{U}^z(x,\mu,\epsilon)\} \\ W^s(\mathcal{M}_\epsilon) &= \{(x,y,z) \,:\, y = \mathcal{S}^y(x,z,\mu,\epsilon), |z| \le \tilde{\epsilon} \ll 1\}. \end{aligned} \tag{4.1}$$

A heteroclinic orbit possibly exists when $W^u(0) \cap W^s(\mathcal{M}_\epsilon)$ nontrivially, or equivalently when the function

$$F(x,\mu,\epsilon) = \mathcal{S}^y(x, \mathcal{U}^z(x,\mu,\epsilon), \mu, \epsilon) - \mathcal{U}^y(x,\mu,\epsilon) \tag{4.2}$$

has a zero.

It is clear that $F(x,0,0) = 0$. If $F_\mu(x,0,0) \neq 0$ for some value of $x \in (0, x_0)$, then an application of the implicit function theorem yields that there exists a smooth function $\mu = \mu(\epsilon)$ such that $F(x,\mu(\epsilon),\epsilon) = 0$. In other words, the condition that $F_\mu \neq 0$ gives that the manifolds $W^u(0)$ and $W^s(\mathcal{M}_\epsilon)$ intersect transversely in (x,y,z,μ)-space, so that their intersection persists for small perturbation. Furthermore, the implicit function theorem yields that

$$\mu'(0) = -\frac{F_\epsilon(x,0,0)}{F_\mu(x,0,0)}. \tag{4.3}$$

Thus, it is possible to get asymptotics for the function $\mu(\epsilon)$ by thoroughly understanding the manner in which the manifolds intersect.

Differentiating (4.2) yields that

$$F_\mu(x,0,0) = (\mathcal{S}^y_\mu - \mathcal{U}^y_\mu + \mathcal{S}^y_z \mathcal{U}^z_\mu)(x,0,0). \tag{4.4}$$

An examination of (2.7) yields that when $\epsilon = 0$ it is unchanged under the transformation $z \to -z$; therefore,

$$\mathcal{S}^y_z(x,0,\mu,0) = 0. \tag{4.5}$$

This gives the simplified expression of F_μ to be

$$F_\mu(x,0,0) = \mathcal{S}^y_\mu(x,0,0,0) - \mathcal{U}^y_\mu(x,0,0). \tag{4.6}$$

In a similiar manner, it is seen that

$$F_\epsilon(x,0,0) = \mathcal{S}^y_\epsilon(x,0,0,0) - \mathcal{U}^y_\epsilon(x,0,0). \tag{4.7}$$

Let the heteroclinic orbit which exists for $\mu = \epsilon = 0$ be denoted by $(\rho(t), u(t), 0)$. Assume that the wave has been translated so that $\rho(0) = x_1$, where $0 < x_1 < x_0$ is the other root of $\lambda(x) = 0$. When linearizing (2.7) about this orbit the variational equations take the form

$$\begin{array}{rcl} \delta x' & = & u\delta x + \rho\delta y \\ \delta y' & = & -\lambda'(\rho)\delta x - 2u\delta y + \delta\mu \\ \delta z' & = & -2u\delta z + (\omega(\rho) - \omega(x_0))\delta\epsilon \\ \delta\mu' & = & 0 \\ \delta\epsilon' & = & 0. \end{array} \tag{4.8}$$

Each of the coordinates δx_i of (4.8) ($x_i = x, y$, etc.) is a 1-form. From these 1-forms exterior products of forms of any degree k can be constructed. The rest of the following discussion will only be concerned with 2-forms $P_{x_i x_j} = \delta x_i \wedge \delta x_j$; however, it can be generalized to any k.

Each 2-form associates to a 2-plane T a number that is the area of the projection of a unit square of T onto the coordinate planes of the two coordinates specified by $P_{x_i x_j}$. In order to evaluate $P_{x_i x_j}$, the following rule is applied. Let N represent a k-dimensional manifold, $k \geq 2$, and let T_pN represent the tangent space to N at a point $p \in N$. Let $\{a_i(p)\}$ represent a basis for T_pN. Suppose that $S(p) \subset T_p(n)$ represents a two-dimensional subspace which is spanned by the vectors $\{a_i(p), a_j(p)\}$ for some $1 \leq i, j \leq k$. These vectors may be thought of as rows to a $2 \times k$ matrix, $A(p)$. If x_i and x_j represent two of the k coordinates of T_pN, then $P_{x_i x_j}$ is the determinant of the 2×2 submatrix obtained by looking at the x_i^{th} and x_j^{th} columns of $A(p)$. Furthermore, the evolution equation for $P_{x_i x_j}$ is given by the product rule, i.e.,

$$P'_{x_i x_j} = \delta x'_i \wedge \delta x_j + \delta x_i \wedge \delta x'_j.$$

Lemma 4.1 *For any* $0 < x < x_0$, $\mathcal{U}^y_\mu(x,0,0) > 0$.

Proof: Set $P_{xy} = \delta x \wedge \delta y$ and $P_{x\mu} = \delta x \wedge \delta\mu$. Using the equations of variation (4.8) it is seen that

$$P'_{xy} = -uP_{xy} + uP_{x\mu}.$$

Two vectors which are tangent to the manifold $W^u(0)$ at a fixed x value are

$$\xi_1 = (\rho', u', 0, 0, 0)^T, \quad \xi_2 = (0, \mathcal{U}^y_\mu, 0, 1, 0)^T.$$

In the above the fact that $\mathcal{U}^z(x,\mu,0) = 0$ is implicitly used. When applied to these vectors

$$P_{x\mu}(\xi_1,\xi_2) = \rho',$$

so that the equation for P_{xy} becomes

$$P'_{xy} = -uP_{xy} + \rho u^2.$$

Since $u = \rho'/\rho$, this equation has the solution

$$\rho(t)P_{xy}(t) = \int_{-\infty}^{t} \rho^2(s)u^2(s)\, ds,$$

so that $P_{xy}(t) > 0$ for all t. Since for a fixed x value

$$P_{xy}(\xi_1,\xi_2) \propto \rho' \mathcal{U}^y_\mu(x,0,0),$$

the fact that $\rho' > 0$ yields the conclusion of the lemma. ∎

Lemma 4.2 *For any* $0 < x < x_0$, $\mathcal{S}^y_\mu(x,0,0,0) < 0$.

Proof: The proof follows that of the previous lemma, and will therefore only be sketched. Set

$$\xi_1 = (\rho', u', 0, 0, 0)^T, \quad \xi_2 = (0, \mathcal{S}^y_\mu, 0, 1, 0)^T.$$

The equation for P_{xy}, when applied to these two vectors, becomes

$$P'_{xy} = -uP_{xy} + \rho u^2,$$

which has the solution

$$\rho(t)P_{xy}(t) = -\int_{t}^{\infty} \rho^2(s)u^2(s)\, ds.$$

Since for any fixed x

$$P_{xy}(\xi_1,\xi_2) \propto \rho' \mathcal{S}^y_\mu(x,0,0,0),$$

the conclusion immediately follows. ∎

Corollary 4.3 *For any* $0 < x < x_0$, $F_\mu(x,0,0) < 0$.

Proof: The result follows immediately from the above lemmas and equation (4.6). ∎

Lemma 4.4 $\mathcal{U}^y_\epsilon(x,0,0) = 0$.

Proof: The proof follows that of Lemma 4.1, and will therefore again only be sketched. Set

$$\xi_1 = (\rho', u', 0, 0, 0)^T, \quad \xi_2 = (0, \mathcal{U}^y_\epsilon, 0, 0, 1)^T.$$

When applied to these two vectors, the equation for P_{xy} is

$$P'_{xy} = -uP_{xy},$$

which has the solution

$$P_{xy}(t) = \frac{C}{\rho(t)}.$$

Since $\lim_{t\to-\infty} P_{xy}(t)$ must be bounded, the fact that $\rho(t) \to 0$ as $t \to -\infty$ implies that the constant C must be zero. Since for fixed x

$$P_{xy}(\xi_1, \xi_2) \propto \rho' \mathcal{U}_\epsilon^y(x, 0, 0),$$

the conclusion of the lemma now follows. ■

The proof of the next lemma mimics those of the previous three lemmas, and will therefore be left to the reader. All that must be kept in mind is that the quantity $P_{xy}(t)$ must approach zero as $t \to \infty$, as $\xi_1(t) \to 0$ and the vector $\xi_2(t)$ remains bounded as $t \to \infty$.

Lemma 4.5 $\mathcal{S}_\epsilon^y(x, 0, 0, 0) = 0.$

Corollary 4.6 $F_\epsilon(x, 0, 0) = 0.$

Proof: The conclusion follows immediately from the previous two lemmas and equation (4.7). ■

As a consequence of Corollaries 4.3 and 4.6, this section can be closed with the following theorem.

Theorem 4.7 *There exists a smooth function $\mu = \mu(\epsilon)$ such that when $0 \leq \epsilon \ll 1$ the manifolds $W^s(\mathcal{M}_\epsilon)$ and $W^u(0)$ have a nontrivial intersection. Furthermore, the function $\mu(\epsilon)$ satisfies*

$$\mu(0) = \mu'(0) = 0,$$

so that

$$\mu(\epsilon) = O(\epsilon^2)$$

Remark 4.8 *Due to the symmetries present in (2.7), when $\mu = \mu(\epsilon)$ it is necessarily true that $W^s(0) \cap W^u(\mathcal{M}_\epsilon)$ is also nontrivial.*

5 $W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon)$ transversely

5.1 Flow from $\mathcal{M}_\epsilon$ to $\{x = \eta\}$

Assume that $\mu = \mu(\epsilon) = \mu_2\epsilon^2 + O(\epsilon^3)$. From the results of Section 4 it is then known that $W^u(0) \cap W^s(\mathcal{M}_\epsilon)$ is nontrivial, as is $W^s(0) \cap W^u(\mathcal{M}_\epsilon)$. Furthermore, on $\mathcal{M}_\epsilon$ there is a connection between the critical points $(x^*(\epsilon), 0, \pm z^*(\epsilon))$. Since there exists a connection between $(0, y_-, z_-)$ and $(0, y_+, z_+)$ in the plane $\{x = 0\}$, it is clear that when $\mu = \mu(\epsilon)$ there is a heteroclinic cycle in the three-dimensional phase space. The goal of this section is to explore the possible solution set as μ is varied from $\mu(\epsilon)$.

Initially, the focus will be on the intersection of $W^u(0)$ and $W^s(\mathcal{M}_\epsilon)$; however, this is not a restriction in any way, as the symmetry described in Proposition 2.1 immediately yields similiar results for the intersection of $W^s(0)$ and $W^u(\mathcal{M}_\epsilon)$. The heteroclinic wave is

singular; however, it can be written in a regular perturbation expansion for $0 \leq x \leq x_0 - \nu$. To set notation, for $0 \leq x \leq x_0 - \nu$ write the wave as

$$\begin{aligned} \rho(t) &= \rho_0(t) + O(\epsilon) \\ u(t) &= u_0(t) + O(\epsilon) \\ \phi(t) &= \epsilon\phi_1(t) + O(\epsilon^2). \end{aligned} \tag{5.1}$$

The following lemma gives a characterization for ϕ_1.

Lemma 5.1 *For $t \in (-\infty, \infty)$ the term ϕ_1 is given by*

$$\phi_1(t) = \beta_1(t)d_1 + \beta_2(t)d_2,$$

where

$$\begin{aligned} \beta_1(t) &= -x_0 \\ \beta_2(t) &= -\frac{1}{4}x_0^3(5 + \tanh(\frac{x_0}{2}t)). \end{aligned}$$

Remark 5.2 *Note that*

$$\frac{\beta_1(t)}{\beta_2(t)} > \frac{1}{2x_0^2}$$

for each t.

Remark 5.3 *Although it is permissable to use ϕ_1 when $t \in (-\infty, T_\nu]$, where $\rho(T_\nu) = x_0 - \nu$, note that $\beta_1(t)$ and $\beta_2(t)$ are well-defined for all t.*

Proof: After substituting the expansion (5.1) into the ODE (2.7) and equating terms it is seen that ϕ_1 satisfies

$$\phi_1' = -2u_0\phi_1 + \omega(\rho_0) - \omega(x_0).$$

Since $u_0 = \rho_0'/\rho_0$, the above equation clearly has the solution

$$\rho_0^2(t)\phi_1(t) = \int_{-\infty}^{t} \rho_0^2(s)(\omega(\rho_0(s)) - \omega(x_0))\,ds.$$

Thus, after using that $\omega(a) = d_1a^2 + d_2a^4$ one finds that

$$\phi_1(t) = \beta_1(t)d_1 + \beta_2(t)d_2,$$

where

$$\beta_i(t) = \frac{1}{\rho_0^2(t)}\int_{-\infty}^{t} \rho_0^2(s)(\rho_0^{2i}(s) - x_0^{2i})\,ds.$$

Substituting the expression for $\rho_0(t)$ given in Proposition 2.2 into the above expression gives the result of the lemma. ∎

Corollary 5.4 *When $\epsilon \neq 0$ the function $\phi(t)$ satisfies*

$$\begin{aligned} \phi(T_\eta) &= -x_0(d_1 + x_0^2d_2 + O(\eta))\epsilon + O(\epsilon^2) \\ \phi(T_\nu) &= -x_0(d_1 + \frac{3}{2}x_0^2d_2 + O(\nu))\epsilon + O(\epsilon^2). \end{aligned}$$

Proof: The result follows immediately from the expansion of the wave given in (5.1) and the above lemma. ∎

It is of interest to understand the behavior of $W^s(\mathcal{M}_\epsilon)$ as it intersects the section B_0^+. For fixed ϵ the intersection of $W^s(\mathcal{M}_\epsilon)$ with B_0^+ forms a curve $y = \mathcal{S}^y(\eta, z, \epsilon)$. Setting T_η to be such that $\rho(T_\eta) = \eta$, it will be desirable to determine $\mathcal{S}^y_z(\eta, \phi(T_\eta), \epsilon)$. In order to find this quantity, the tangent space to the manifold must be tracked.

In the four-dimensional phase space the manifold $W^s(\mathcal{M}_\epsilon)$ is three-dimensional and can be written as the graph

$$W^s(\mathcal{M}_\epsilon) = \{(x, y, z, \epsilon) \,:\, \eta \leq x \leq x_0, y = \mathcal{S}^y(x, z, \epsilon)\}. \tag{5.2}$$

Over the underlying wave, for each fixed x the manifold's tangent space is spanned by the three vectors

$$\begin{aligned} \xi_1 &= (\rho', u', \phi', 0)^T \\ \xi_2 &= (0, \mathcal{S}^y_z, 1, 0)^T \\ \xi_3 &= (0, \mathcal{S}^y_\epsilon, 0, 1)^T. \end{aligned} \tag{5.3}$$

Since $W^s(\mathcal{M}_\epsilon)$ is a three-dimensional manifold, it will be necessary to use three-forms to properly track its tangent space. In particular, the forms

$$P_{xy\epsilon} = \delta x \wedge \delta y \wedge \delta\epsilon$$

and

$$P_{xz\epsilon} = \delta x \wedge \delta z \wedge \delta\epsilon$$

will be used. Note that for fixed x

$$P_{xy\epsilon}(\xi_1, \xi_2, \xi_3) = \rho' \mathcal{S}^y_z,$$

and in particular

$$\mathcal{S}^y_z(\eta, \phi(T_\eta), \epsilon) \propto \frac{P_{xy\epsilon}(T_\eta)}{\rho'(T_\eta)}. \tag{5.4}$$

Thus, it will be desirable to compute $P_{xy\epsilon}$ at $x = \eta$.

After linearizing about the wave for $\epsilon \neq 0$ the variational equations become

$$\begin{aligned} \delta x' &= u\delta x + \rho\delta y \\ \delta y' &= -\lambda'(\rho)\delta x - 2u\delta y + 2\phi\delta z + (2\mu_2\epsilon + O(\epsilon^2))\delta\epsilon \\ \delta z' &= \epsilon\omega'(\rho)\delta x - 2\phi\delta y - 2u\delta z + (\omega(\rho) - \omega(x_0) - \epsilon\sigma)\delta\epsilon \\ \delta\epsilon' &= 0. \end{aligned} \tag{5.5}$$

The ODE for $P_{xy\epsilon}$ is then

$$P'_{xy\epsilon} = -uP_{xy\epsilon} + 2\phi P_{xz\epsilon}, \tag{5.6}$$

which, using the fact that $u = \rho'/\rho$, has the solution

$$\rho(t)P_{xy\epsilon}(t) = (x_0 - \nu)P_{xy\epsilon}(T_\nu) - 2\int_t^{T_\nu} \rho(s)\phi(s)P_{xz\epsilon}(s)\,ds. \tag{5.7}$$

It is clear that the evolution of $P_{xz\epsilon}$ must be understood in order to understand the evolution of $P_{xy\epsilon}$. Towards this end is the following proposition.

Proposition 5.5 *When $\epsilon = 0$,*

$$P_{xz\epsilon}(t) = x_0^2 \frac{u_0(t)}{\rho_0(t)}.$$

Proof: In order to prove the proposition, it will first be shown that the vector ξ_2 is given by

$$\xi_2(t) = (0, 0, x_0^2/\rho_0^2(t), 0)^T.$$

Recall that the vector ξ_2 is formed by taking the derivative of the manifold $W^s(\mathcal{M}_\epsilon)$ with respect to z. Since the flow has the symmetry $(x, y, z, t) \to (x, y, -z, t)$ when $\epsilon = 0$, it is clear that the first two components of ξ_2 must be zero. From the variational equations the third component satisfies the ODE

$$\delta z' = -2u_0 \delta z,$$

which has the solution

$$\delta z(t) = C/\rho_0^2(t).$$

The constant C can be chosen so that

$$\lim_{t\to\infty} \delta z(t) = 1,$$

from which arises the characterization of $\xi_2(t)$.

Since $\xi_1(t) = (\rho_0'(t), u_0'(t), 0, 0)$, the conclusion of the proposition now immediately follows, as

$$P_{xz\epsilon}(\xi_1, \xi_2, \xi_3) = \begin{vmatrix} \rho_0' & 0 & * \\ 0 & x_0^2/\rho_0^2 & * \\ 0 & 0 & 1 \end{vmatrix}. \qquad \blacksquare$$

Recall that T_ν and T_η are defined by

$$\rho(T_\eta) = \eta, \quad \rho(T_\nu) = x_0 - \nu.$$

For $t \in [T_\eta, T_\nu]$ the three-form $P_{xz\epsilon}$ is given by the regular perturbation expansion

$$P_{xz\epsilon}(t) = x_0^2 \frac{u_0(t)}{\rho_0(t)} + O(\epsilon). \tag{5.8}$$

Substitution of this expression into (5.7) along with the expansion for ϕ given in Lemma 5.1 yields that

$$\rho_0(t) P_{xy\epsilon}(t) = (x_0 - \nu) P_{xy\epsilon}(T_\nu) - 2\epsilon x_0^2 \int_t^{T_\nu} u_0(s)(\beta_1(s)d_1 + \beta_2(s)d_2)\, ds + O(\epsilon^2). \tag{5.9}$$

In order to finish the calculation, $P_{xy\epsilon}(T_\nu)$ must be determined.

As it has already been seen,

$$P_{xy\epsilon}(T_\nu) = \rho'(T_\nu) \mathcal{S}_z^y(x_0 - \nu, \phi(T_\nu), \epsilon);$$

thus, the quantity $\mathcal{S}_z^y(x_0 - \nu, \phi(T_\nu), \epsilon)$ must be calculated. Since ν is small, this can be determined by understanding the variation of the linear approximation to the manifold $W^s(\mathcal{M}_\epsilon)$. Since $\mathcal{M}_\epsilon \subset W^s(\mathcal{M}_\epsilon)$ it is necessarily true that

$$\mathcal{S}^y(\mathcal{M}_x(z, \epsilon), z, \epsilon) = \mathcal{M}_y(z, \epsilon). \tag{5.10}$$

Recall that Lemma 3.6 states that the manifold $\mathcal{M}_\epsilon$ satisfies

$$\partial_z \mathcal{M}_y(z,\epsilon) = O(\zeta^2).$$

As such, due to (5.10) the slow manifold has no effect on $W^s(\mathcal{M}_\epsilon)$, at least up to $O(\epsilon)$. Therefore, in order to understand the variation of $W^s(\mathcal{M}_\epsilon)$ it it enough to determine how the vectors which are tangent to $W^s(\mathcal{M}_\epsilon)$ at $\mathcal{M}_\epsilon$ vary with respect to z and ϵ. Recall Proposition 3.8, which states that the tangent vectors to $W^s(\mathcal{M}_\epsilon)$ at $\mathcal{M}_\epsilon$ are given by

$$(-1, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}),$$

where the quantities v_i are defined in Proposition 3.8. In order to determine $\mathcal{S}_z^y$, it is then enough to calculate $\partial_z(v_2/v_1)$, where these quantities are defined in Proposition 3.8. A simple calculation yields

$$-\partial_z \frac{v_2}{v_1} = \epsilon \frac{\omega'(x_0)}{x_0 \lambda'(x_0)} + O(\zeta^2),$$

from which it is seen that

$$\mathcal{S}_z^y(x_0-\nu, \phi(T_\nu), \epsilon) = \epsilon(\frac{\omega'(x_0)}{x_0\lambda'(x_0)} + O(\nu)) + O(\zeta^2).$$

Since $\rho'(T_\nu) = O(\nu)$ one then has that

$$P_{xy\epsilon}(T_\nu) = O(\nu)\epsilon + O(\zeta^2)$$

Upon combining the above results, and using the fact that $\beta_i(t)$ is bounded for $i = 1, 2$ and for all t, it is seen that

$$\rho_0(t)P_{xy\epsilon}(t) = (\alpha_1(t)d_1 + \alpha_2(t)d_2 + O(\nu))\epsilon + O(\epsilon^2), \tag{5.11}$$

where

$$\alpha_i(t) = -2x_0^2 \int_t^\infty u_0(s)\beta_i(s)\, ds \tag{5.12}$$

for $i = 1, 2$.

Proposition 5.6 *When evaluated at $t = T_\eta$,*

$$\begin{aligned} \alpha_1(T_\eta) &= -2x_0^3 \ln \frac{\eta}{x_0} \\ \alpha_2(T_\eta) &= -2x_0^5 \ln \frac{\eta}{x_0} + \frac{1}{2}x_0^5(1 - \frac{\eta^2}{x_0^2}). \end{aligned}$$

Proof: The expression for $\alpha_1(T_\eta)$ only will be proved, as the proof for $\alpha_2(T_\eta)$ is similiar. For the rest of this proof set $k = x_0/2$.

Using the expression for $\rho_0(t)$ given in Proposition 2.2 it is not difficult to see that

$$\begin{aligned} u_0(t) &= \rho_0'(t)/\rho_0(t) \\ &= \frac{x_0}{4}\frac{\operatorname{sech}^2(kt)}{1+\tanh(kt)}. \end{aligned}$$

Since, by Lemma 5.1, $\beta_1(t) = -x_0$, it is clear that

$$\begin{aligned}\alpha_1(t) &= x_0^3 \int_{kt}^{\infty} \frac{\mathrm{sech}^2(s)}{1+\tanh(s)}\, ds \\ &= x_0^3(\ln(2) - \ln(\frac{2\rho_0^2(t)}{x_0^2}).\end{aligned}$$

The last part of the above expression comes from the fact that

$$1 + \tanh(kt) = 2\rho_0^2(t)/x_0^2.$$

Since $\rho_0(T_\eta) = \eta$, upon simplification the expression for $\alpha_1(T_\eta)$ follows. ∎

Remark 5.7 *Note that*

$$\frac{\alpha_2(T_\eta)}{\alpha_1(T_\eta)} > x_0^2,$$

with

$$\frac{\alpha_2(T_\eta)}{\alpha_1(T_\eta)} = x_0^2 + \frac{1}{4}x_0^2(-\ln\frac{\eta}{x_0})^{-1} + O(\eta^2)$$

for $0 < \eta \ll 1$.

Now that $\alpha_i(T_\eta)$ has been calculated for $i = 1, 2$, the following lemma has essentially been proved. All that is now necessary is to recall (5.4), the fact that $0 < \rho'(T_\eta) = O(\eta)$, and the fact that $y_+ = x_0/2$.

Lemma 5.8 *The manifold* $W^s(\mathcal{M}_\epsilon)$ *satisfies the estimate*

$$\mathcal{S}_z^y(\eta, \phi(T_\eta), \epsilon) \propto \frac{1}{\eta^2}(\alpha_1 d_1 + \alpha_2 d_2 + O(\nu))\epsilon + O(\epsilon^2),$$

where

$$\begin{aligned}\alpha_1 &= -4x_0^2 \ln\frac{\eta}{x_0} \\ \alpha_2 &= -4x_0^4\ln\frac{\eta}{x_0} + x_0^4(1 - \frac{\eta^2}{x_0^2}).\end{aligned}$$

Remark 5.9 *Note that* α_1/α_2 *satisfies the same estimate as that given in Remark 5.7.*

Corollary 5.10 *The manifold* $W^u(\mathcal{M}_\epsilon)$ *satisfies the estimate*

$$\mathcal{U}_z^y(\eta, -\phi(T_\eta), \epsilon) = \mathcal{S}_z^y(\eta, \phi(T_\eta), \epsilon).$$

Proof: The result follows immediately from the symmetry outlined in Proposition 2.1. ∎

5.2 Flow from $\{x = \eta\}$ to $\{y = y_- + \eta\}$

Recall the transversality argument given in Section 4. It was shown how the manifolds $W^u(0)$ and $W^s(\mathcal{M}_\epsilon)$ vary with respect to μ along the wave when $\epsilon = 0$. Due to the smoothness of the flow, these estimates persist for ϵ and μ sufficiently small. Furthermore, as a by-product of the symmetry discussed in Proposition 2.1, the manner in which $W^s(0)$ and $W^u(\mathcal{M}_\epsilon)$ vary with respect to μ is also understood.

Specifically, the following is known. Set

$$\begin{array}{rcl} W^u(\mathcal{M}_\epsilon) \cap \{x = \eta\} & = & \{(y, z, \mu, \epsilon) : y = \mathcal{U}^y(z, \mu, \epsilon)\} \\ W^s(0) \cap \{x = \eta\} & = & \{(y, z, \mu, \epsilon) : y = \mathcal{S}^y(\mu, \epsilon), z = \mathcal{S}^z(\mu, \epsilon)\}. \end{array} \tag{5.13}$$

As a consequence of Proposition 2.1 and Lemmas 4.1 and 4.2 it can be concluded that

$$\mathcal{S}^y_\mu(0,0) < 0, \quad \mathcal{U}^y_\mu(0,0,0) > 0; \tag{5.14}$$

furthermore, these estimates hold for ϵ and μ sufficiently small. When $\mu = \mu(\epsilon)$ the manifolds intersect, so that

$$\mathcal{S}^y(\mu, \epsilon) = \mathcal{U}^y(\mathcal{S}^z(\mu, \epsilon), \mu, \epsilon). \tag{5.15}$$

Thus, when ϵ is fixed, by the estimate (5.14) for $\mu > \mu(\epsilon)$ the manifold $W^s(0)$ is below $W^u(\mathcal{M}_\epsilon)$, while for $\mu < \mu(\epsilon)$ the configuration is reversed.

Recall Lemma 3.2, which gives a description of the curves $\mathcal{C}_0 \cdot t_0^\pm$ and $\mathcal{C}_{\pm\infty} \cdot t_{\pm\infty}$. Specifically, it is of interest here to contemplate the nature of $\mathcal{C}_0 \cdot t_0^-$ and $\mathcal{C}_{-\infty} \cdot t_{-\infty}$, both of which are contained in B_0^- and centered upon

$$p_0^s = W^s(0) \cap B_0^-. \tag{5.16}$$

The set

$$\mathcal{C}_0^- = \mathcal{C}_{-\infty} \cdot t_{-\infty} \cup p_0^s \cup \mathcal{C}_0 \cdot t_0^- \tag{5.17}$$

defineds a curve in B_0^-; in fact, for $\epsilon\omega(x_0) \neq 0$ it is a two-armed logarithmic spiral centered upon p_0^s. When $\mu = \mu(\epsilon)$, $W^u(\mathcal{M}_\epsilon)$ intersects $\mathcal{C}_0^-$ at p_0^s; indeed, the intersection is transverse. Thus, as μ is varied from $\mu(\epsilon)$, $W^u(\mathcal{M}_\epsilon)$ continues to intersect $\mathcal{C}_0^-$; however, the intersection no longer occurs at p_0^s. This implies that $W^u(\mathcal{M}_\epsilon)$ intersects either $\mathcal{C}_{-\infty} \cdot t_{-\infty}$ or $\mathcal{C}_0 \cdot t_0^-$. From (5.14) and Lemma 3.2 it can be concluded that if $|\mu - \mu(\epsilon)|$ is $O(\epsilon^n)$ for $n \geq 3$ but is not exponentially small in ϵ, then

$$\begin{array}{rcl} \mu > \mu(\epsilon) & \Longrightarrow & W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \cdot t_0^- \neq \emptyset \\ \mu < \mu(\epsilon) & \Longrightarrow & W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_{-\infty} \cdot t_{-\infty} \neq \emptyset \end{array} \tag{5.18}$$

(see Figure 3). Furthermore, there exists only one point in each of the intersections. If $0 < |\mu - \mu(\epsilon)| \leq O(e^{-c/\epsilon})$, then there exists a finite number of points in both $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \cdot t_0^-$ and $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_{-\infty} \cdot t_{-\infty}$, with the number of points increasing to infinity as $\mu \to \mu(\epsilon)$.

For the rest of this subsection it will be assumed that $|\mu - \mu(\epsilon)|$ is not exponentially small in ϵ. This is done to clarify the following arguments. In addition, it will be assumed that $\mu \geq \mu(\epsilon)$; however, this is not a necessary restriction and the below arguments can be modified to draw conclusions if $\mu < \mu(\epsilon)$.

Since $\mu > \mu(\epsilon)$, by (5.18) $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \cdot t_0^-$ is nonempty, which immediately implies that $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0$ is nonempty. Due to symmetry outlined in Proposition 2.1 this yields not only that $W^s(\mathcal{M}_\epsilon) \cap \mathcal{C}_0$ is nonempty, but

$$W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 = W^s(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \subset W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon).$$

Thus, there is a connection between $(x^*(\epsilon), 0, z^*(\epsilon))$ and $(x^*(\epsilon), 0, -z^*(\epsilon))$ which passes "near" (to be made more precise later) the plane $\{x = 0\}$. The goal of the rest of this subsection is to understand the nature in which $W^u(\mathcal{M}_\epsilon)$ and $W^s(\mathcal{M}_\epsilon)$ intersect at $\mathcal{C}_0$.

Now that the orientation of the manifolds $W^u(\mathcal{M}_\epsilon)$ and $W^s(\mathcal{M}_\epsilon)$ is understood as they intersect the plane $\{x = \eta\}$, it is desirable to understand their behavior under the flow as they pass near the critical points $(0, y_-, z_-)$ and $(0, y_+, z_+)$, respectively. The goal of this subsection is to show that under a suitable restriction the passage near these critical points does not effect their orientation. It will be sufficient to track the manifold $W^u(\mathcal{M}_\epsilon)$, as the symmetry present in (2.7) allows one to then draw an immediate conclusion regarding the behavior of $W^s(\mathcal{M}_\epsilon)$.

First, suppose that μ is such that

$$0 < O(\epsilon^{n+1}) \leq |\mu - \mu(\epsilon)| \leq O(\epsilon^n) \tag{5.19}$$

for some $n \geq 3$. As it will be seen, this restriction on μ will guarantee that the spiralling action near the critical points on $\{x = 0\}$ will be unseen by $W^u(\mathcal{M}_\epsilon)$ as it passes near $\{x = 0\}$. Since $n \geq 2$, the perturbation of μ given in (5.19) will not effect the asymptotic estimates given in Lemma 5.8 and Corollary 5.10. Using the asymptotics for $\mathcal{U}^y$, it will be desirous to look at the image of the curve

$$x = \eta, \quad y = y_0 + \epsilon c z, \tag{5.20}$$

where $|z| \leq \eta$, $y_0 = O(\epsilon^n)$, and

$$\operatorname{sgn}(c) = \operatorname{sgn}(\mathcal{U}^y_z).$$

The point y_0 describes the difference between μ and $\mu(\epsilon)$, and by supposition is therefore positive.

It will be advantageous to look at the flow near the critical point in polar coordinates. Upon setting

$$y = r\sin\theta, \quad z = r\cos\theta$$

the curve (5.20) can be parametrically described as

$$\begin{aligned} r_0(s) &= \sqrt{y_0^2 + 2c\epsilon s + (1 + \epsilon^2 c^2)s^2} \\ \theta_0(s) &= \operatorname{Tan}^{-1}(\frac{y_0 + \epsilon c s}{s}), \end{aligned} \tag{5.21}$$

where $|s| \leq \eta$. The linear flow near $(0, y_-, z_-)$ satisfies the ODE

$$\begin{aligned} x' &= -ax \\ r' &= 2ar \\ \theta' &= \epsilon b, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} a &= \sqrt{-\lambda(0)} + O(\epsilon^2) \\ b &= \frac{\omega(x_0)}{\sqrt{-\lambda(0)}} + O(\epsilon). \end{aligned}$$

Thus, upon using (5.21) as an initial condition to (5.22) it is seen that the behavior of $\mathcal{U}^y_z$ is well approximated by the solution formula

$$\begin{aligned} x(t) &= \eta e^{-at} \\ r(t) &= r_0(s)e^{2at} \\ \theta(t) &= \theta_0(s) + \epsilon b t. \end{aligned} \tag{5.23}$$

It is now of interest to determine the nature of (5.21) as the flow forces it to intersect the cylinder $\{r = \eta\}$. The time of flight, t_f, is defined by $r(t_f) = \eta$. Using (5.23), this yields that

$$t_f = \frac{1}{2a} \ln \frac{\eta}{r_0(s)}.$$

Thus, under the linear flow (5.23) there is the mapping from $\{x = \eta\}$ to $\{r = \eta\}$ given by

$$\begin{pmatrix} \eta \\ r_0(s) \\ \theta_0(s) \end{pmatrix} \to \begin{pmatrix} \sqrt{\eta}\sqrt{r_0(s)} \\ \eta \\ \theta_0(s) + \epsilon \frac{b}{2a} \ln \frac{\eta}{r_0(s)} \end{pmatrix} \tag{5.24}$$

for $|s| \leq \eta$. Set $x(s)$ to be the first component and $\theta(s)$ to be the third component of the vector on the right hand side of the above equation. Given the asymptotic expansion for $\mathcal{U}_z^y$, it is then of interest to perform an expansion of $x(s)$ and $\theta(s)$ around $s = 0$. This step is taken because it is only of interest to determine the behavior of the curve near where $W^u(\mathcal{M}_\epsilon)$ intersects $\mathcal{C}_0 \cdot t_0^-$. First, a Taylor expansion yields that

$$\begin{aligned} r_0(s) &= y_0 + \epsilon c s + O(w^2) \\ \theta_0(s) &= \frac{\pi}{2} - \frac{1}{y_0} s + O(w^2), \end{aligned}$$

where $w = \epsilon s / y_0$ and it has been used that $y_0 > 0$. Plugging this expansion into (5.24) and performing another Taylor expansion then gives that

$$\begin{aligned} x(s) &= \sqrt{\eta}(\sqrt{y_0} + \frac{\epsilon c}{2\sqrt{y_0}} s + O(w^2)) \\ \theta(s) &= \frac{\pi}{2} + \epsilon \frac{b}{2a} \ln \frac{\eta}{y_0} - \frac{1}{y_0}(1 + \epsilon^2 \frac{ac}{2b}) s + O(w^2). \end{aligned} \tag{5.25}$$

In rectangular coordinates the right hand side of (5.24) is given by $x(s)$, $y(s) = \eta \sin\theta(s)$, and $z(s) = \eta \cos\theta(s)$. Using the expansion (5.25) it is then seen that

$$\begin{aligned} x(0) &= \sqrt{\eta}\sqrt{y_0}, & x'(0) &= \sqrt{\eta}\frac{\epsilon c}{2\sqrt{y_0}} \\ y(0) &= \eta \sin\theta(0), & y'(0) &= -\eta(\frac{1}{y_0}(1 + \epsilon^2 \frac{ac}{2b})) \cos\theta(0) \\ z(0) &= \eta \cos\theta(0), & z'(0) &= \eta(\frac{1}{y_0}(1 + \epsilon^2 \frac{ac}{2b})) \sin\theta(0). \end{aligned} \tag{5.26}$$

It is clear that the curve given by the right hand side of (5.24) can be parameterized by z for s sufficiently near zero. Using the expansion coefficients given in (5.26) it is then seen that $x = x(z)$ satisfies the relation

$$\frac{dx}{dz} = \frac{\epsilon c \sqrt{y_0}}{2\sqrt{\eta}(1 + \epsilon^2 \frac{ac}{2b})) \sin\theta(0)}. \tag{5.27}$$

Since

$$\theta(0) = \frac{\pi}{2} + \epsilon \frac{b}{2a} \ln \frac{\eta}{y_0},$$

this means that for $y_0 = O(\epsilon^n)$,

$$\theta(0) = \frac{\pi}{2} + O(n\epsilon \ln \frac{1}{\epsilon}).$$

Substituting this estimate into (5.27) and applying (5.26) yields the following lemma.

Lemma 5.11 *Suppose that $\mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 2$. The intersection of $W^u(\mathcal{M}_\epsilon)$ with the plane $\{y = y_- + \eta\}$ is a parametric curve given by $(x, z) = (x(s), z(s))$. The curve satisfies the estimates*

$$\begin{array}{ll} a. & x(0) = O(\epsilon^{n/2}) \\ b. & \left.\dfrac{dx}{dz}\right|_{s=0} = O(\epsilon^{(n+2)/2}) \\ c. & sgn\,(\dfrac{dx}{dz}) = sgn\,(\mathcal{U}_z^y(\eta, -\phi(T_\eta), \mu(\epsilon), \epsilon)). \end{array}$$

Remark 5.12 *If $W^u(\mathcal{M}_\epsilon)$ intersects the plane $\{y = y_- - \eta\}$, then the conclusion of the lemma holds, with part c. being changed to*

$$sgn\,(\frac{dx}{dz}) = -sgn\,(\mathcal{U}_z^y(\eta, -\phi(T_\eta), \mu(\epsilon), \epsilon)).$$

As an immediate consequence of the symmetry of the ODE one gets the following corollary.

Corollary 5.13 *Suppose that $\mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 2$. The intersection of $W^s(\mathcal{M}_\epsilon)$ with the plane $\{y = y_+ - \eta\}$ is a parametric curve given by $(x, z) = (x(s), z(s))$. The curve satisfies the estimates*

$$\begin{array}{ll} a. & x(0) = O(\epsilon^{n/2}) \\ b. & \left.\dfrac{dx}{dz}\right|_{s=0} = O(\epsilon^{(n+2)/2}) \\ c. & sgn\,(\dfrac{dx}{dz}) = sgn\,(\mathcal{S}_z^y(\eta, \phi(T_\eta), \mu(\epsilon), \epsilon)). \end{array}$$

Remark 5.14 *If $W^u(\mathcal{M}_\epsilon)$ intersects the plane $\{y = y_+ + \eta\}$, then the conclusion of the lemma holds, with part c. being changed to*

$$sgn\,(\frac{dx}{dz}) = -sgn\,(\mathcal{U}_z^y(\eta, \phi(T_\eta), \mu(\epsilon), \epsilon)).$$

5.3 Flow from $\{y = y_- + \eta\}$ to $\{y = 0\}$

Now that the passage of the manifold $W^u(\mathcal{M}_\epsilon)$ near the critical point $(0, y_-, z_-)$ is understood, it is necessary to understand its behavior under the flow near $\{x = 0\}$. The previous lemma states that the manifold is $O(\epsilon^{n/2})$ near $\{x = 0\}$ when it intersects the plane $\{y = y_- + \eta\}$. Since $x' = xy$ and $y < 0$ in the region of interest, it is clear that manifold stays within $O(\epsilon^{n/2})$ of $\{x = 0\}$ until it intersects the plane $\{y = 0\}$. Thus, it is expected that the flow on $\{x = 0\}$ dominates the behavior of the manifold. The goal of this subsection is to make this intuition rigorous.

After the manifold $W^u(\mathcal{M}_\epsilon)$ has intersected the plane $\{y = y_- + \eta\}$ it can be written as the graph

$$W^u(\mathcal{M}_\epsilon) = \{(x, y, z) \,:\, x = \mathcal{U}^x(y, z, \epsilon), y_- + \eta \leq y \leq y_+ - \eta, |z| \leq \eta\}. \tag{5.28}$$

The previous lemma gives an expression for $\mathcal{U}_z^x$ along the wave when $y = y_- + \eta$. The goal is to calculate $\mathcal{U}_z^x(0, 0, \epsilon)$. In order to accomplish this, it will be necessary to once again use differential forms.

The vector to be tracked has the initial condition

$$\xi = (\mathcal{U}_z^x(y_- + \eta, z, \epsilon), 0, 1)^T,$$

where the z-coordinate is taken to be such that the vector is over the underlying solution. The variational equations are

$$\begin{aligned}\delta x' &= u\delta x + \rho\delta y \\ \delta y' &= -\lambda'(\rho)\delta x - 2u\delta y + 2\phi\delta z \\ \delta z' &= \epsilon\omega'(\rho)\delta x - 2\phi\delta y - 2u\delta z.\end{aligned} \tag{5.29}$$

Projectivize the equations by setting

$$a = \delta x/\delta z, \quad b = \delta y/\delta z. \tag{5.30}$$

It is not difficult to check that $\delta z(\xi) = O(1)$ for $y_- + \eta \le y \le 0$, so that the above quantities are well-defined in the region of interest. The equations for a and b are given by

$$\begin{aligned}a' &= (-u + 2\phi b)a + \rho b - \epsilon\omega'(\rho)a^2 \\ b' &= -\lambda'(\rho)a + 2\phi(1 + b^2) - \epsilon\omega'(\rho)ab,\end{aligned} \tag{5.31}$$

with the initial conditions being

$$\begin{aligned}&a(0) = \mathcal{U}_z^x(u \cdot 0, \phi \cdot 0, \epsilon) \\ &b(0) = 0.\end{aligned}$$

Without loss of generality, assume that the z-coordinate of $W^u(0) \cap B_0^-$, $-\phi(T_\eta)$, is such that $-\phi(T_\eta) > 0$. By Corollary 5.4, this implies that

$$d_1 + x_0^2 d_2 > 0. \tag{5.32}$$

Let $(\rho, u, \phi) \cdot t \in W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon)$ represent the orbit connecting $(x^*(\epsilon), 0, \pm z^*(\epsilon))$, and let it be normalized so that $u \cdot 0 = y_- + \eta$. It is known that this orbit intersects $\mathcal{C}_0$, as $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \cdot t_0^-$ is nontrivial. Therefore, there is a $T_0 > 0$ such that $u \cdot T_0 = \phi \cdot T_0 = 0$. In addition, there exists a $T_1 < 0$ such that $\rho \cdot T_1 = \eta$.

The claim is that $\phi \cdot t > 0$ for $t \in [T_1, T_0]$. To see that this is true, consider the following argument. The z-coordinate of $W^u(\mathcal{M}_\epsilon) \cap \mathcal{C}_0 \cdot t_0^-$, $\phi \cdot T_1$, is within $O(\epsilon^3)$, $n \ge 3$, of $-\phi(T_\eta)$; therefore, since $-\phi(T_\eta) > 0$ is $O(\epsilon)$, $\phi \cdot T_1$ is also positive. Following the argument of the previous subsection, it is not difficult to see that $\phi \cdot t > 0$ for $t \in [T_1, 0]$. Since $\phi \cdot 0 > 0$, as a consequence of Proposition 3.1 it is necessarily true that $\phi \cdot t > 0$ for $t \in [0, T_0]$ (recall that, by Lemma 5.11, $\rho \cdot 0 = O(\epsilon^{n/2})$).

As a consequence of Corollary 5.10 and Lemma 5.11 it is known that

$$\begin{aligned}&\operatorname{sgn}(\mathcal{U}_z^x(u \cdot 0, \phi \cdot 0, \epsilon)) = \operatorname{sgn}(\alpha_1 d_1 + \alpha_2 d_2) \\ &\mathcal{U}_z^x(u \cdot 0, \phi \cdot 0, \epsilon) = O(\epsilon^{(n+2)/2}).\end{aligned}$$

In what follows, assume that

$$\alpha_1 d_1 + \alpha_2 d_2 > 0. \tag{5.33}$$

Since α_1/α_2 is close to $1/x_0^2$ (for an exact description see Lemma 5.8), this is not that much more a restriction than that already imposed by (5.32).

Under this hypothesis, it is then being assumed that $a(0) > 0$, with $a(0) = O(\epsilon^{(n+2)/2})$. It is of interest to calculate $a(T_0)$, as

$$\mathcal{U}_z^x(0, 0, \epsilon) \propto a(T_0). \tag{5.34}$$

Towards this end is the following lemma.

Lemma 5.15 *$a(t) \le O(\epsilon^{n/2})$ for $t \in [0, T_0]$.*

Proof: First, since $\rho' = \rho u$,

$$\rho(t) = \rho(0) e^{\int_0^t u(s)\,ds},$$

so that $\rho(t) = O(\rho(0)) = O(\epsilon^{n/2})$ for $t \in [0, T_0]$. In addition, since $\delta z(\xi)$ is $O(1)$ for $y_- + \eta \le y \le 0$, and $\delta x(\xi)$ and $\delta y(\xi)$ satisfy the linear ODE (5.29), it is necessarily true that a and b are $O(1)$ for $t \in [0, T_0]$. These two facts will be used extensively in what follows.

Set

$$\begin{aligned} f(t) &= e^{\int_0^t u(s) - 2\phi(s)b(s)\,ds} \\ &= \frac{\rho(t)}{\rho(0)} e^{-2\int_0^t \phi(s)b(s)\,ds}. \end{aligned}$$

Since $b(t) = O(1)$ and $\rho(t) = O(\rho(0))$, it is clear that $f(t) = O(1)$ for $t \in [0, T_0]$. Solving (5.31), it is seen that $a(t)$ is given by the integral formula

$$f(t)a(t) = a(0) + \int_0^t f(s)\rho(s)b(s)\,ds - \epsilon \int_0^t f(s)\omega'(\rho(s))a^2(s)\,ds. \tag{5.35}$$

Using the estimates on f and ρ, and assuming that a and b are $O(1)$, it can then be concluded that

$$f(t)a(t) = a(0) + O(\epsilon^{n/2}) \cdot t + O(\epsilon^{(n+2)/2}) \cdot t.$$

Since $a(0) = O(\epsilon^{(n+2)/2})$, the conclusion follows. ■

It will be desirable to revise the above estimate. In order to do so, the following lemma is needed.

Lemma 5.16 *$b(t) = O(\epsilon)$ for $t \in [0, T_0]$.*

Proof: Since $b(0) = 0$, the function $b(t)$ satisfies the integral equation

$$b(t) = 2\int_0^t \phi(s)(1 + b^2(s))\,ds - \int_0^t (\lambda'(\rho(s))a(s) + \epsilon\omega'(\rho(s))a(s)b(s))\,ds. \tag{5.36}$$

Assuming that $b(t) = O(1)$, which was justified in the proof of the above lemma, using the conclusion of the above lemma, and using the fact that $\phi(t) = O(\epsilon)$, one gets that

$$b(t) = O(\epsilon) \cdot t + O(\epsilon^n) \cdot t.$$

Since $T_0 = O(1)$, the conclusion now follows. ■

A more careful examination of (5.36) yields the following result for $b(t)$, which in turn can be used to improve upon the estimate made upon $a(t)$ in Lemma 5.15. Using the estimates provided in the above two lemmas, it is clear that

$$b' = 2\phi(1 + b^2) + O(\epsilon^n),$$

which can be solved to get

$$b(t) = \tan(\int_0^t 2\phi(s)\,ds) + O(\epsilon^n) \cdot t.$$

Substituting this expression into (5.35), and again using the above two lemmas, yields

$$f(t)a(t) = a(0) + \int_0^t f(s)\rho(s)\tan(\int_0^s 2\phi(r)\,dr)\,ds + O(\epsilon^{3n/2}) \cdot t.$$

Note that the second term on the right-hand side of the above equation is $O(\epsilon^{(n+2)/2})$, as is $a(0)$. Furthermore, this term is positive, as $\phi(t) > 0$ for $t \in [0, T_0]$. Since, by supposition, $a(0) > 0$, it is now clear that

$$f(T_0)a(T_0) > a(0),$$

and furthermore, $a(T_0) = O(a(0))$.

Using (5.34), the proof of the following lemma is now complete. While the lemma was proved only for the case of both $\phi \cdot 0 > 0$ and $a(0) > 0$, it can easily be modified in the event that both quantities are negative.

Lemma 5.17 *Suppose that $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 3$. Further suppose that*

$$sgn\,(\alpha_1 d_1 + \alpha_2 d_2) = sgn\,(d_1 + x_0^2 d_2),$$

where the α_i's are defined in Lemma 5.8. Then

$$\begin{aligned} &sgn\,(\mathcal{U}_z^x(0,0,\epsilon)) = sgn\,(\alpha_1 d_1 + \alpha_2 d_2) \\ &\mathcal{U}_z^x(0,0,\epsilon) = O(\epsilon^{(n+2)/2}). \end{aligned}$$

By the symmetries present in (2.7) one arrives at the following corollary.

Corollary 5.18 *Let the hypotheses of Lemma 5.17 be satisfied. Then*

$$sgn\,(\mathcal{S}_z^x(0,0,\epsilon)) = -sgn\,(\mathcal{U}_z^x(0,0,\epsilon)).$$

Using the time of flight estimates given in Subsection 5.2, one finally gets the following theorem.

Theorem 5.19 *Suppose that $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 3$. Further suppose that the parameters d_1 and d_2 are chosen so that*

$$sgn\,(\alpha_1 d_1 + \alpha_2 d_2) = sgn\,(d_1 + x_0^2 d_2),$$

or where the α_i's are defined in Lemma 5.8. Then the manifolds $W^u(\mathcal{M}_\epsilon)$ and $W^s(\mathcal{M}_\epsilon)$ intersect transversely at $\{y = 0\}$, with the transversality being $O(\epsilon^{(n+2)/2})$. Furthermore, the time the resultant trajectory spends in the region $0 \leq x \leq \eta$ is given by

$$T_f = O(2n \ln \frac{1}{\epsilon}).$$

5.4 Completion of argument

Using the notation of the previous section, near the invariant plane $\{x = 0\}$ write

$$\begin{aligned} W^u(\mathcal{M}_\epsilon) &= \{(x,y,z,\epsilon) \,:\, x = \mathcal{U}^x(y,z,\epsilon), y_- + \eta \leq y \leq y_+ - \eta\} \\ W^s(\mathcal{M}_\epsilon) &= \{(x,y,z,\epsilon) \,:\, x = \mathcal{S}^x(y,z,\epsilon), y_- + \eta \leq y \leq y_+ - \eta\}. \end{aligned}$$

By Theorem 5.19 it is known that if $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$ for $n \geq 3$ and if the pair (d_1, d_2) satisfy the hypotheses of Lemma 5.17, then there exists a $(\tilde{x}(\epsilon), 0, 0) \in \mathcal{C}_0$, $\tilde{x}(\epsilon) = O(\epsilon^{n/2})$, such that

$$\begin{aligned}
&\mathcal{U}^x(0,0,\epsilon) = \mathcal{S}^x(0,0,\epsilon)\\
&\mathrm{sgn}\,((\mathcal{U}^x_z - \mathcal{S}^x_z)(0,0,\epsilon)) = \mathrm{sgn}\,(\alpha_1 d_1 + \alpha_2 d_2)\\
&(\mathcal{U}^x_z - \mathcal{S}^x_z)(0,0,\epsilon) = O(\epsilon^{(n+2)/2}).
\end{aligned}$$

Since the manifolds intersect transversely at $(\tilde{x}(\epsilon), 0, 0)$, there is an orbit, $(\rho(t), u(t), \phi(t))$, which connects the two critical points $(x^*(\epsilon), 0, \pm z^*(\epsilon)) \in \mathcal{M}_\epsilon$. Let this trajectory by translated so that $(\rho(0), u(0), \phi(0)) = (\tilde{x}(\epsilon), 0, 0) \in \mathcal{C}_0$.

Since the manifolds $W^u(\mathcal{M}_\epsilon)$ and $W^s(\mathcal{M}_\epsilon)$ intersect transversely at $\{y = 0\}$, they intersect transversely everywhere along the orbit. The next goal is to understand the transversality as $W^u(\mathcal{M}_\epsilon)$ intersects $B^+_{x_0}$. Define

$$\begin{aligned}
\xi_1 &= (\rho'(0), u'(0), \phi'(0))^T\\
\xi_2 &= (\mathcal{U}^x_z(\tilde{x}(\epsilon), 0, 1)^T\\
\xi_3 &= (\mathcal{S}^x_z(\tilde{x}(\epsilon), 0, 1)^T.
\end{aligned} \tag{5.37}$$

It is clear that $\xi_2 \cdot t \in TW^u(\mathcal{M}_\epsilon)$ and $\xi_3 \cdot t \in TW^s(\mathcal{M}_\epsilon)$ for all t, and that $\xi_1 \cdot t \in TW^u(\mathcal{M}_\epsilon) \cap TW^s(\mathcal{M}_\epsilon)$ for all t. The three-form P_{xyz} will be used to gain an understanding as to how the flow carries these vectors. Using (5.29), the three-form P_{xyz} satisfies Abel's formula

$$P'_{xyz} = -3uP_{xyz},$$

which, using the fact that $u = \rho'/\rho$, has the solution

$$P_{xyz}(t) = \frac{\rho^3(0)}{\rho^3(t)} P_{xyz}(0). \tag{5.38}$$

When applied to the tangent vectors ξ_i defined in (5.37),

$$P_{xyz}(0) = (\lambda(0) + O(\epsilon^2))(\mathcal{U}^x_z - \mathcal{S}^x_z)(0,0,\epsilon); \tag{5.39}$$

thus, after substituting into (5.38) one gets

$$P_{xyz}(t) = \frac{\rho^3(0)}{\rho^3(t)}(\lambda(0) + O(\epsilon^2))(\mathcal{U}^x_z - \mathcal{S}^x_z)(0,0,\epsilon). \tag{5.40}$$

Now define $\tilde{T}_\nu$ such that $\rho(\tilde{T}_\nu) = x_0 - \nu$. As they intersect the section $\{x = x_0 - \nu\}$ the manifolds $W^u(\mathcal{M}_\epsilon)$ and $W^s(\mathcal{M}_\epsilon)$ are given by the curves

$$\begin{aligned}
W^u(\mathcal{M}_\epsilon) \cap B^+_{x_0} &= \{(y, z, \epsilon) \,:\, y = \mathcal{U}^y(z, \epsilon)\}\\
W^s(\mathcal{M}_\epsilon) \cap B^+_{x_0} &= \{(y, z, \epsilon) \,:\, y = \mathcal{S}^y(z, \epsilon)\}.
\end{aligned}$$

In addition, there exists a $\tilde{z}(\epsilon)$ such that

$$\mathcal{U}^y(\tilde{z}(\epsilon), \epsilon) = \mathcal{S}^y(\tilde{z}(\epsilon), \epsilon).$$

In order to understand the transversality of the manifolds at $\{x = x_0 - \nu\}$ it is necessary to compute $(\mathcal{U}^y_z - \mathcal{S}^y_z)(\tilde{z}(\epsilon), \epsilon)$. Towards this end, set

$$\begin{aligned}
\tilde{\xi}_2 &= (0, \mathcal{U}^y_z(\tilde{z}(\epsilon), \epsilon), 1)^T\\
\tilde{\xi}_3 &= (0, \mathcal{S}^y_z(\tilde{z}(\epsilon), \epsilon), 1)^T.
\end{aligned}$$

It is clear that

$$\begin{array}{rcl} P_{xyz}(\tilde{T}_\nu) & \propto & P_{xyz}(\xi_1 \cdot \tilde{T}_\nu, \tilde{\xi}_2, \tilde{\xi}_3) \\ & = & \rho'(\tilde{T}_\nu)(\mathcal{U}_z^y - \mathcal{S}_z^y)(\tilde{z}(\epsilon), \epsilon). \end{array} \tag{5.41}$$

Since $\rho'(\tilde{T}_\nu) > 0$ is $O(\nu)$, upon substituting (5.40) into (5.41) one gets

$$\begin{array}{rcl} (\mathcal{U}_z^y - \mathcal{S}_z^y)(\tilde{z}(\epsilon), \epsilon) & \propto & \rho^3(0)\lambda(0)(\mathcal{U}_z^x - \mathcal{S}_z^x)(0, 0, \epsilon) \\ & = & A\epsilon^{2n+1} + O(\epsilon^{2n+2}), \end{array} \tag{5.42}$$

where $\mathrm{sgn}\,(A) = -\mathrm{sgn}\,(\alpha_1 d_1 + \alpha_2 d_2)$ (recall that $\lambda(0) < 0$). The argument for the following lemma is now complete.

Lemma 5.20 *Let $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 3$. There exists a $\tilde{z}(\epsilon)$, with $|\tilde{z}(\epsilon) - \phi(T_\nu)| = O(\epsilon^{n-1})$, such that*

1. $\mathcal{U}^y(\tilde{z}(\epsilon), \epsilon) = \mathcal{S}^y(\tilde{z}(\epsilon), \epsilon)$
2. $sgn\,((\mathcal{U}_z^y - \mathcal{S}_z^y)(\tilde{z}(\epsilon), \epsilon)) = -sgn\,(\alpha_1 d_1 + \alpha_2 d_2)$
3. $(\mathcal{U}_z^y - \mathcal{S}_z^y)(\tilde{z}(\epsilon), \epsilon) = O(\epsilon^{2n+1})$.

Remark 5.21 *To paraphrase, the manifold $W^u(\mathcal{M}_\epsilon)$ intersects $W^s(\mathcal{M}_\epsilon)$ transversely at the section $\{x = x_0 - \nu\}$, with the transversality being $O(\epsilon^{2n+1})$ for $0 < \mu - \mu(\epsilon) = O(\epsilon^n)$.*

6 Existence of solitary waves

6.1 Existence of bright solitary waves

A bright solitary wave is characterized by

$$\lim_{t \to \pm\infty} x(t) = 0;$$

hence, in order to have such a wave it is necessary that $W^u(0) \cap W^s(0)$ be nontrivial. Due to the symmetry described in Proposition 2.1, it is enough to show that $W^u(0) \cap \mathcal{C}_{x_0} \cdot t_{x_0}$, and hence $W^u(0) \cap \mathcal{C}_{x_0}$.

Recall Lemma 3.11, which states that $\mathcal{C}_{x_0} \cdot t_{x_0}$ is $C^1 - O(e^{-c/\epsilon})$ close to $W^s(\mathcal{M}_\epsilon)$ for points $p \in \mathcal{C}_{x_0}$ which take $O(1/\epsilon)$ time to exit B_{x_0}. Under a time reversal, the flow on $\mathcal{M}_\epsilon$ for $z = O(\epsilon)$ satisfies

$$z' = \epsilon^2\sigma + O(\epsilon^3).$$

Therefore, $\mathcal{C}_{x_0} \cdot t_{x_0}$ will be close to $W^s(\mathcal{M}_\epsilon)$ in a region where $z = O(\epsilon)$ with $\mathrm{sgn}\,(z) = \mathrm{sgn}\,(\sigma)$.

Now recall Corollary 5.4, which states that when $\mu = \mu(\epsilon)$ that $\phi(T_\nu)$, the z-coordinate of $W^u(0) \cap B_{x_0}^+$, is given by

$$\phi(T_\nu) = -x_0(d_1 + \frac{3}{2}x_0^2 d_2 + O(\nu))\epsilon + O(\epsilon^2).$$

Thus, by the comments of the previous paragraph, if

$$(d_1 + \frac{3}{2}x_0^2 d_2)\sigma < 0,$$

then $W^u(0) \cap W^s(\mathcal{M}_\epsilon)$ is C^1-$O(e^{-c/\epsilon})$ close to the curve $\mathcal{C}_{x_0} \cdot t_{x_0}$ (see Figures 4 and 5). Using the facts that $\mathcal{C}_{x_0} \cdot t_{x_0}$ is C^1-$O(e^{-c/\epsilon})$ close to $W^s(\mathcal{M}_\epsilon)$ and that $W^u(0)$ intersects $W^s(\mathcal{M}_\epsilon)$ transversely yield the following lemma.

Lemma 6.1 *Suppose that*

$$(d_1 + \frac{3}{2}x_0^2 d_2)\sigma < 0$$

(see Figure 6). There exists a $\mu_h(\epsilon) < \mu(\epsilon)$, with $\mu(\epsilon) - \mu_h(\epsilon) = O(e^{-c/\epsilon})$, such that when $\mu = \mu_h(\epsilon)$, then $W^u(0) \cap W^s(0)$ is nontrivial.

Remark 6.2 *The fact that $\mu(\epsilon) - \mu_h(\epsilon) > 0$ is a consequence of Lemmas 4.1 and 4.2.*

The solution described in the above lemma can be thought of as a 1-pulse solution. It is characterized by the fact $W^u(0)$ intersects the plane $\{y = 0\}$ at exactly one point. An N-pulse solution is characterized by both $W^u(0) \cap W^s(0)$ being nontrivial and $W^u(0)$ intersecting $\{y = 0\}$ at N distinct points. Given the 1-pulse solution, it is natural to inquire as to the existence of N-pulses. Fortunately, this question has been studied in the work of Kapitula and Maier-Paape [**?**]. In order to quote the results stated in that paper, the next lemma must first be proved.

Lemma 6.3 *Let $0 < \epsilon \ll 1$ be fixed. Let $z_h(\mu)$ represent the z-coordinate of $W^u(0) \cap \{y = 0\}$. Then*

$$\begin{aligned} &1. \qquad z_h(\mu_h(\epsilon)) = 0 \\ &2. \qquad \frac{d}{d\mu} z_h(\mu_h(\epsilon)) \neq 0. \end{aligned}$$

Proof: The first part of the conclusion follows immediately from the fact that when $\mu = \mu_h(\epsilon)$, $W^u(0) \cap \mathcal{C}_{x_0} \neq \emptyset$.

For fixed ϵ the set $W^u(0) \cap B_{x0}^+$ yields a curve parameterized by μ, i.e.,

$$W^u(0) \cap B_{x0}^+ = \{(y, z, \mu) \,:\, y = y^u(\mu), z = z^u(\mu)\}.$$

Due to Lemma 4.1 it is known that

$$\frac{d}{d\mu} y^u(0) > 0,$$

so that

$$\frac{d}{d\mu} y^u(\mu_h(\epsilon)) > 0, \tag{6.1}$$

as $\mu_h(\epsilon) = O(\mu(\epsilon)) = O(\epsilon^2)$.

Let $p = (x_0 - \nu, y^u(\mu_h(\epsilon)), z^u(\mu_h(\epsilon))) \in W^u(0) \cap B_{x0}^+$, and let $T_h > 0$ be such that $p \cdot T_h \in \mathcal{C}_{x_0}$. Since $z' = -\epsilon^2\sigma + O(\epsilon^3)$ and $z^u(\mu_h(\epsilon)) = O(\epsilon)$, it is necessarily true that $T_h = O(1/\epsilon)$. Therefore, as a consequence of the Exchange Lemma with Exponentially Small Error [**?**], the transversality described by (6.1) gets transferred into a transversality condition in the slow direction. Since the slow direction near $\mathcal{M}_\epsilon$ is described by z, this means that

$$\frac{d}{d\mu} z_h(\mu_h(\epsilon)) \neq 0. \qquad \blacksquare$$

With the above lemma in hand, it is now possible to state Theorem 1.7 in Kapitula and Maier-Paape [**?**].

Theorem 6.4 *Let $0 < \epsilon \ll 1$, and suppose that*

$$(d_1 + \frac{3}{2}x_0^2 d_2)\sigma < 0.$$

For each $N \geq 2$ there exists a bi-infinite sequence $\{\mu_k^N\}$ such that when $\mu = \mu_k^N$ there is an N-pulse solution to (2.7). The N-pulse is such that $x(t)$ is even in t. Furthermore,

$$|\mu_k^N - \mu_h(\epsilon)| = O(e^{-c|k|/\epsilon})$$

as $|k| \to \infty$.

Remark 6.5 *The estimate on $|\mu_k^N - \mu_h(\epsilon)|$ is not explicitly provided in Theorem 1.7 of Kapitula and Maier-Paape; however, it is implicit in the proof of that theorem.*

Remark 6.6 *Although one can discuss the existence of N-pulses which are odd in t, this will not be done here. For a more complete description of the dynamical behavior for μ near $\mu_h(\epsilon)$, the interested reader should consult [**?**].*

6.2 Existence of dark solitary waves

A dark solitary wave is characterized by

$$\lim_{t\to\pm\infty} (x, y, z) \cdot t \in \mathcal{M}_\epsilon;$$

hence, in order to have such a wave it is necessary that both $W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon) \neq \emptyset$ and there exist critical points in $\mathcal{M}_\epsilon$. By Proposition 3.4, the existence of the critical points is guaranteed if

$$(d_1 + 2x_0^2 d_2)\sigma < 0. \tag{6.2}$$

By Lemma 5.20, $W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon) \neq \emptyset$ if

$$\begin{aligned} &0 < \mu - \mu(\epsilon) = O(\epsilon^n), \quad n \geq 3 \\ &(d_1 + x_0^2 d_2)(\alpha_1 d_1 + \alpha_2 d_2) > 0, \end{aligned} \tag{6.3}$$

where, by Lemma 5.8, the coefficients α_i satisfy

$$\frac{\alpha_2}{\alpha_1} = x_0^2 + \frac{1}{4}x_0^2(-\ln\frac{\eta}{x_0})^{-1} + O(\eta^2).$$

Furthermore, the manifolds intersect transversely, with the transversality being $O(\epsilon^{2n+1})$ (Lemma 5.20).

An N-circuit solution is a dark solitary wave whose trajectory passes near $\{x = 0\}$ N times. If (6.2) and (6.3) are satisfied, then for $0 < \epsilon \ll 1$ there is a 1-circuit solution. The goal is to show that there exist N-circuits for $2 \leq N < N(\epsilon)$, where $N(\epsilon) \to \infty$ as $\epsilon \to 0$. This will be accomplished through the use of the Exchange Lemma with Exponentially Small Error (ELESE) (Jones et al [**?**]).

In order for the ELESE to apply, it must first be shown that $W^u(\mathcal{M}_\epsilon) \cap W^s(\mathcal{M}_\epsilon)$ transversely. This task was accomplished in the series of lemmas leading to Lemma 5.20. The ELESE then applies to all those points in

$$\mathcal{C}_u^+ = W^u(\mathcal{M}_\epsilon) \cap B_{x_0}^+$$

which take $O(1/\epsilon)$ time to exit B_{x_0}.

Set

$$\mathcal{C}_s^+ = W^s(\mathcal{M}_\epsilon)B_{x_0}^+.$$

The z-coordinate of $\mathcal{C}_u^+ \cap \mathcal{C}_s^+$, say $\hat{z}$, is given by

$$\hat{z} = -x_0(d_1 + \frac{3}{2}x_0^2 d_2 + O(\nu))\epsilon + O(\epsilon^2),$$

i.e., $|\hat{z} - \phi(T_\nu)| = O(\epsilon^2)$. This is due to the assumption that $\mu - \mu(\epsilon) = O(\epsilon^n)$ for some $n \geq 3$. Set

$$\mathcal{C}_u^- = W^u(\mathcal{M}_\epsilon) \cap B_{x_0}^-, \quad \mathcal{C}_s^- = W^s(\mathcal{M}_\epsilon) \cap B_{x_0}^-.$$

Due to the symmetry inherent in the ODE, not only is $\mathcal{C}_u^- \cap \mathcal{C}_s^- \neq \emptyset$, the z-coordinate of the intersection is given by $-\hat{z}$. Now, recall that for $|z| = O(\epsilon)$ the flow on $\mathcal{M}_\epsilon$ is given by

$$z' = -\epsilon^2\sigma + O(\epsilon^3)$$

(equation 3.39). Thus, if

$$(d_1 + \frac{3}{2}x_0^2 d_2)\sigma < 0 \tag{6.4}$$

the flow on $\mathcal{M}_\epsilon$ is from $\hat{z}$ to $-\hat{z}$; otherwise, it is in the opposite direction.

Assume that (6.4) holds. Since $z' = O(\epsilon^2)$, it takes a time of $O(1/\epsilon)$ for trajectories to traverse from $\hat{z}$ to $-\hat{z}$. Let $\mathcal{C}_u^+ \cdot T_u^+$ represent the intersection of the curve $\mathcal{C}_u^+$ with the section $B_{x_0}^-$ as it is carried by the flow generated by the ODE (2.7). By the ELESE the curve $\mathcal{C}_u^+ \cdot T_u^+$ will be C^1-$O(e^{-c/\epsilon})$ close to $\mathcal{C}_u^-$ at $-\hat{z}$. Since $\mathcal{C}_u^- \cap \mathcal{C}_s^-$ transversely at $-\hat{z}$, with the order of transversality being $O(\epsilon^{2n+1})$, this implies that $\mathcal{C}_u^+ \cdot T_u^+ \cap \mathcal{C}_s^-$ transversely exponentially close to $-\hat{z}$. In other words, upon passing through B_{x_0} the manifold $W^u(\mathcal{M}_\epsilon)$ becomes oriented so that it once again intersects $W^s(\mathcal{M}_\epsilon)$ transversely. Given the existence of a 1-circuit solution, the existence of a 2-circuit has now been proven.

The above argument can be repeated to show the existence of N-circuit solutions for each $N \geq 2$. Given an $\epsilon > 0$, the maximal N, say $N(\epsilon)$, will be such that $N(\epsilon) \to \infty$ as $\epsilon \to 0$. This is due to the fact that upon passage through B_{x_0}, the curve $\mathcal{C}_u^+ \cdot T_u^+$ is C^1-$O(e^{-c/\epsilon})$ close to $\mathcal{C}_u^-$ at $-\hat{z}$. The following theorem has now been proven.

Theorem 6.7 *Suppose that $0 < \epsilon \ll 1$, and suppose that (6.2), (6.3), and (6.4) hold (see Figure 7). Then there exists an $N(\epsilon) > 1$, with $N(\epsilon) \to \infty$ as $\epsilon \to 0$, such that N-circuit solutions exist for $1 \leq N < N(\epsilon)$.*

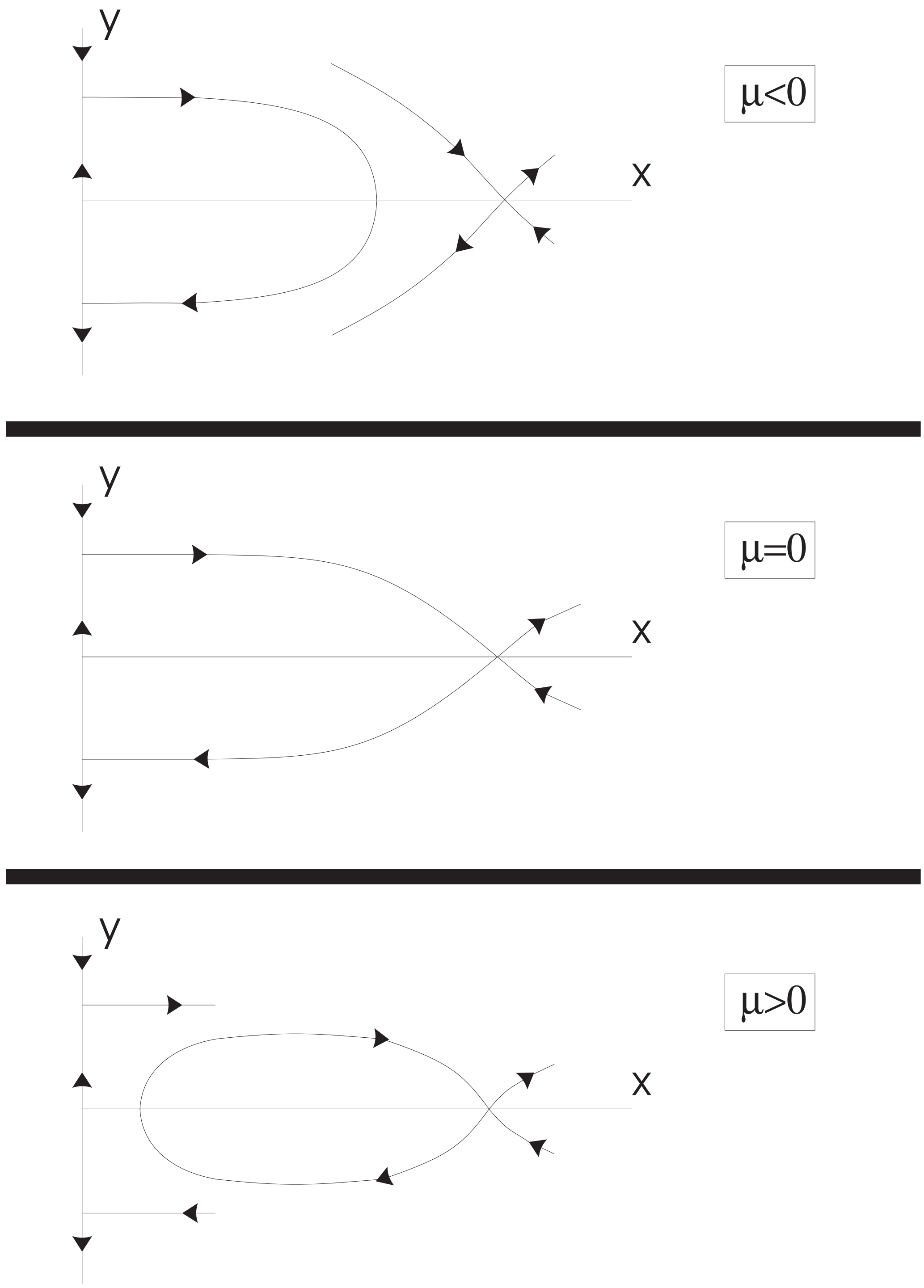

Figure 1: Flow on $\{z = 0\}$ when $\epsilon = 0$

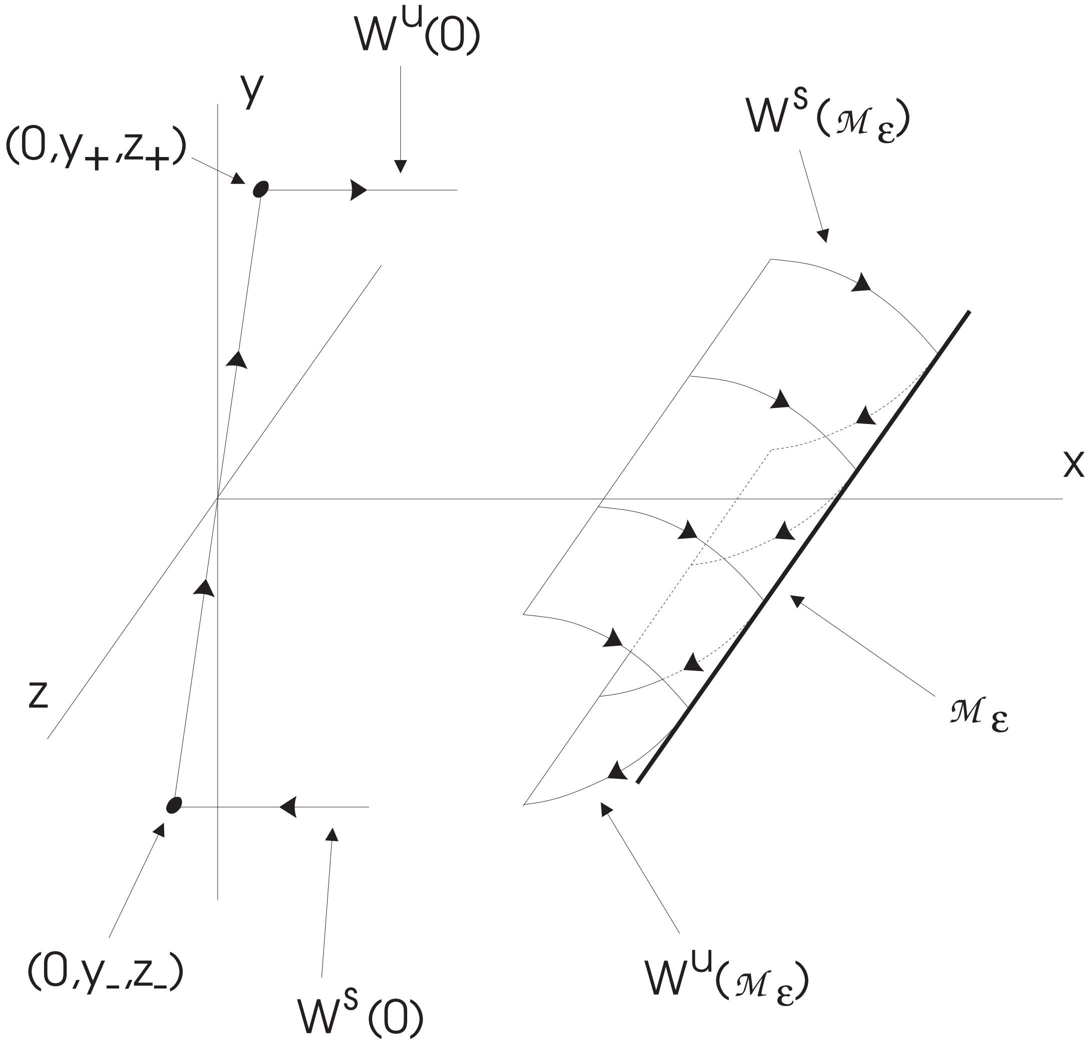

Figure 2: The phase space geometry

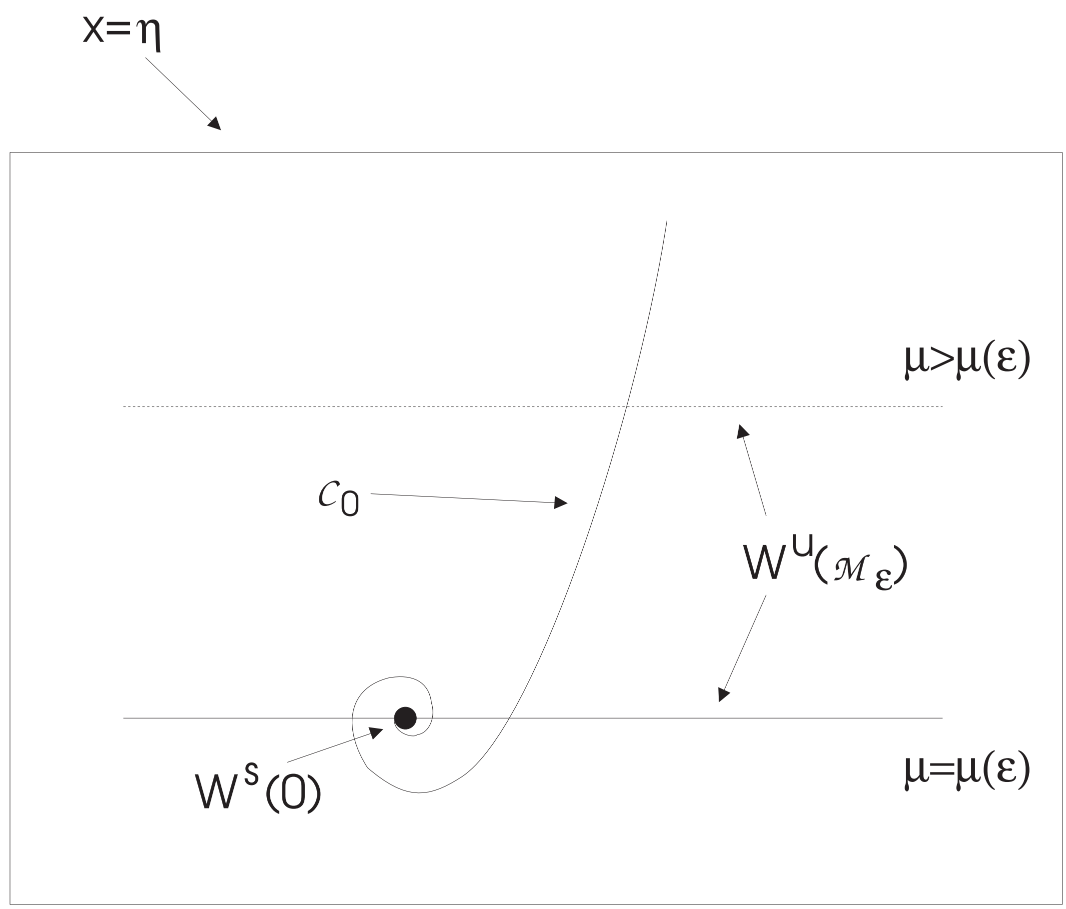

Figure 3: Existence of dark solitary waves

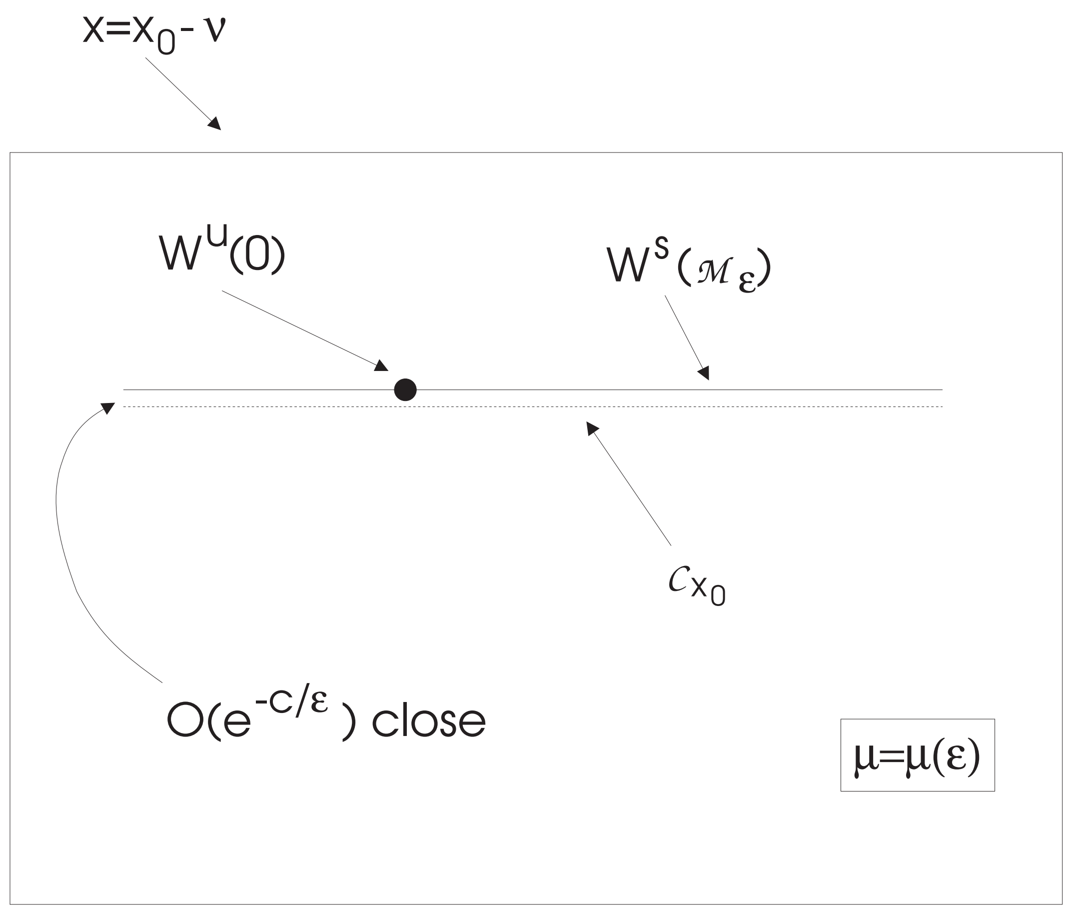

Figure 4: Existence of bright solitary waves

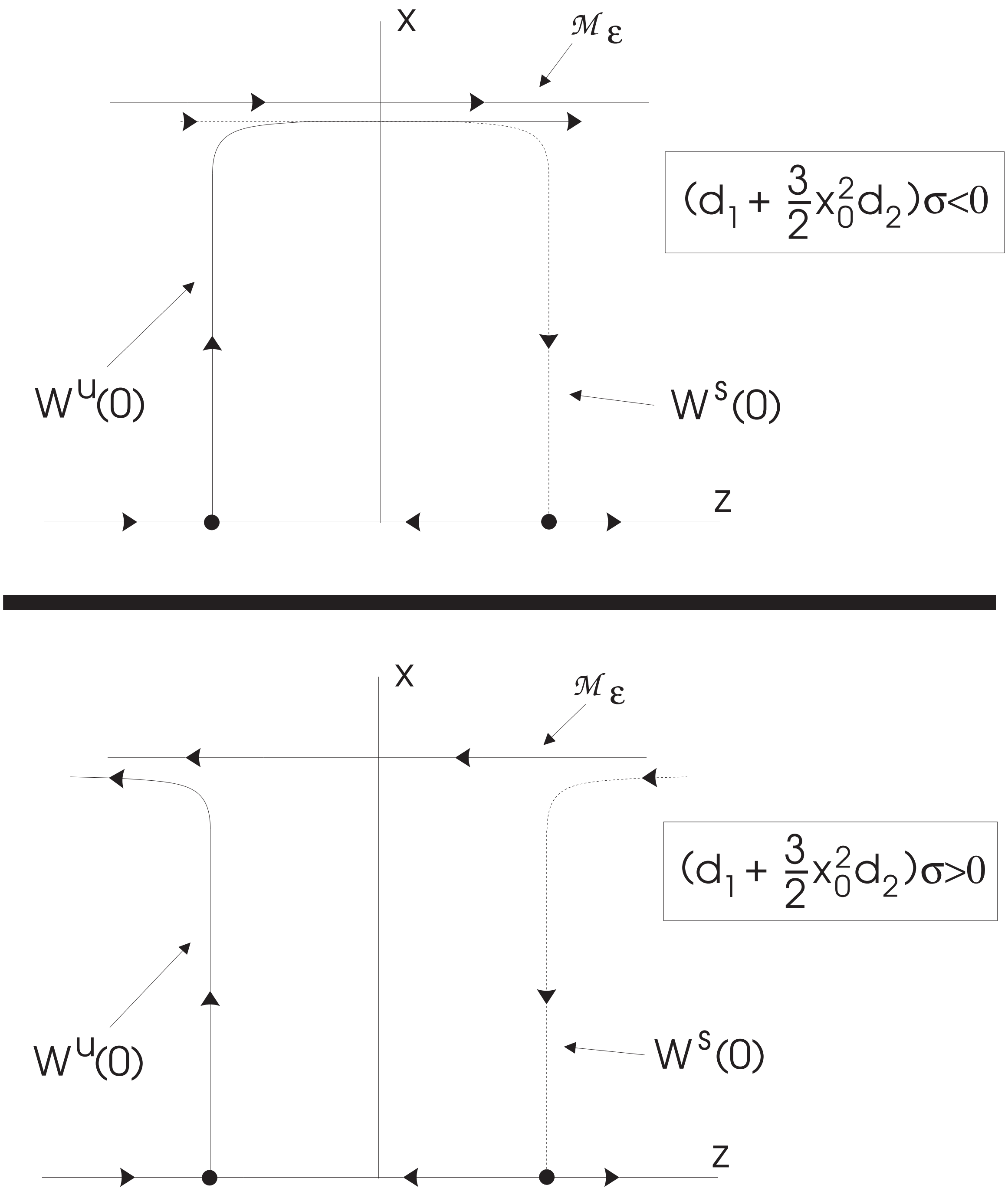

Figure 5: Projection of flow on $\{y = 0\}$

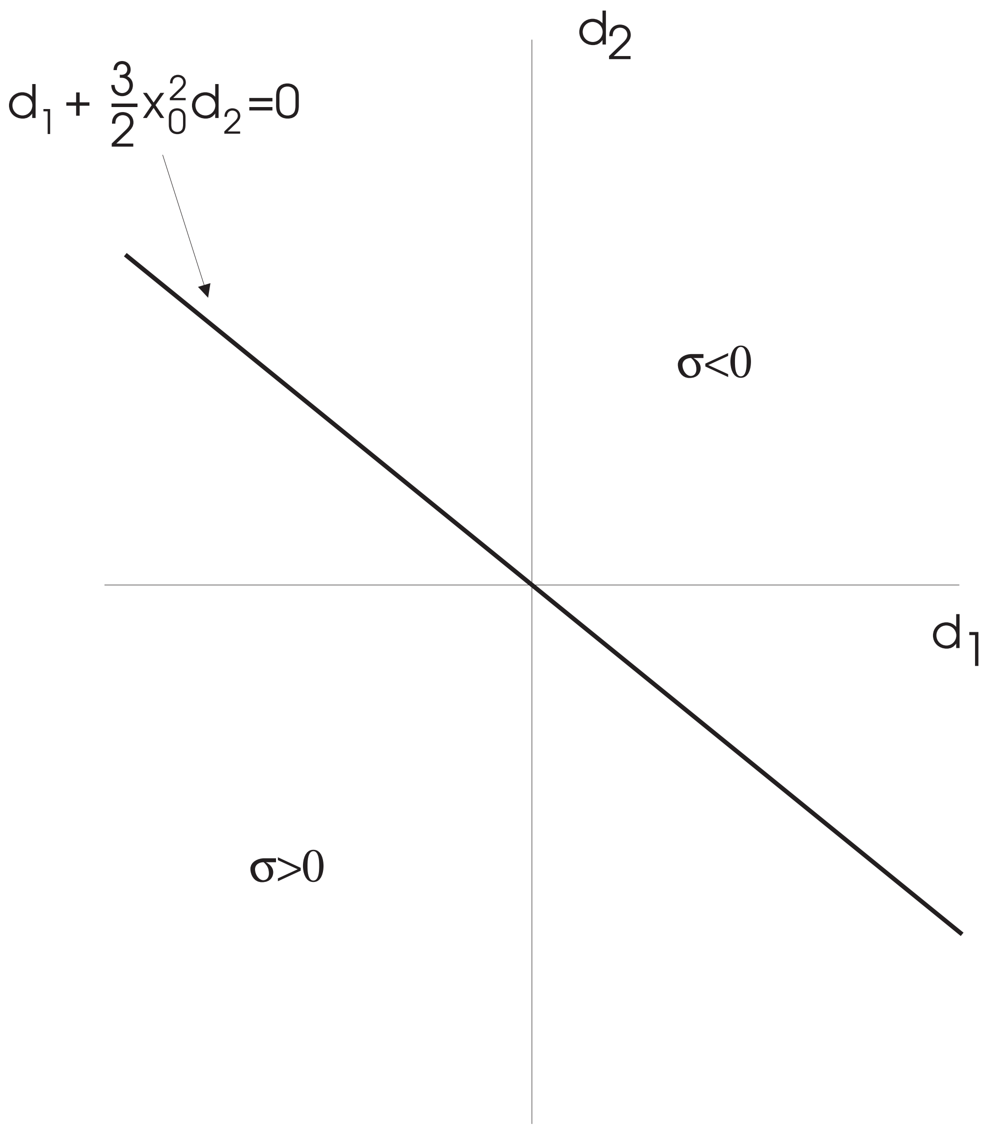

Figure 6: Parameter regime for bright solitary waves

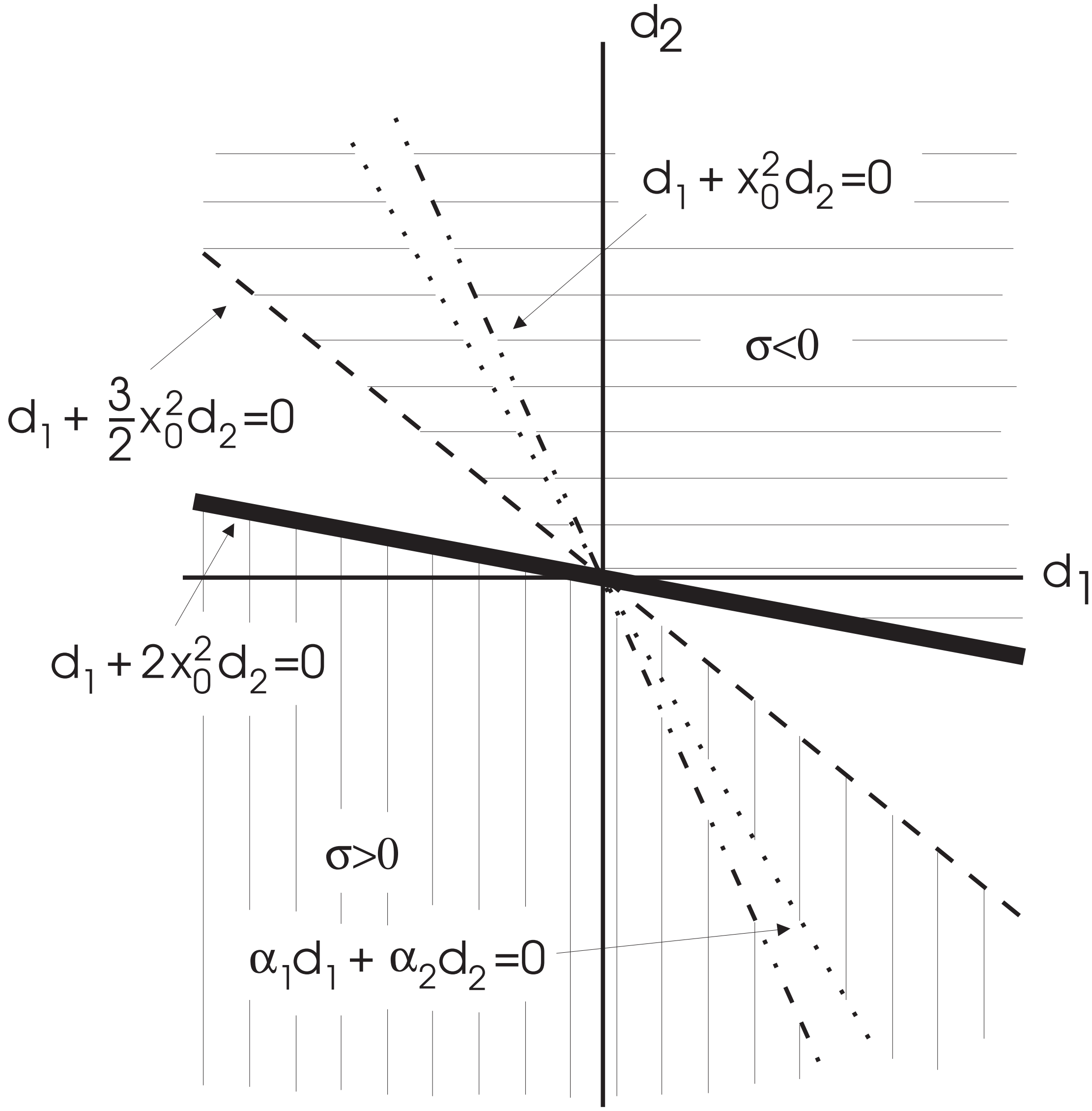

Figure 7: Parameter regime for dark solitary waves